

UNIQUENESS OF FORM EXTENSIONS AND DOMINATION OF SEMIGROUPS

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ABSTRACT. In this article, we present a new method to treat uniqueness of form extensions in a rather general setting including various magnetic Schrödinger forms. The method is based on the theory of ordered Hilbert spaces and the concept of domination of semigroups. We review this concept in an abstract setting and give a characterization in terms of the associated forms. Then we use it to prove a theorem that transfers uniqueness of form extension of a dominating form to that of a dominated form. This result is applied to two classes of examples: magnetic Schrödinger forms on graphs and on domains in Euclidean space.

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INTRODUCTION

In many situations in mathematics and physics one is given a Laplace type operator on a domain of smooth functions and looks for self-adjoint extensions. Existence of such self-adjoint extensions follows from general theory of forms. Indeed, there always exists an extension with Dirichlet boundary condition and an extension with Neumann boundary condition. Accordingly, these are the most common extensions. Now, it may well be that these two extensions agree and the question whether this happens is of quite some interest [6, 17, 20, 22, 42]. More generally, there is the question of essential self-adjointness i.e. uniqueness of a self-adjoint extension, and substantial attention has been devoted to proving essential self-adjointness for Laplacians on manifolds [3, 24, 26, 41] and on graphs [4, 11, 12, 19, 21, 48].

On the structural level, the question whether the Laplacian with Dirichlet and the Laplacian with Neumann boundary conditions agree is connected to an additional feature of the Laplacian viz that it may generate a Markov semigroup. More specifically, in typical situation all self-adjoint extensions of the Laplacian which generate a Markov semigroup can be shown to lie between the Laplacian with Dirichlet boundary conditions and the Laplacian with Neumann boundary conditions (see [8] for open subsets of Euclidean space, [10] for locally finite graphs and [40] for recent results dealing with general Dirichlet forms). Thus, equality of these two boundary conditions then amounts to uniqueness of a self-adjoint extension generating a Markov process. This phenomenon is known as Markov uniqueness. Clearly, it is of substantial interest in any operator theoretic treatment of Markov processes.

The corresponding questions can be asked for Laplacians on functions as well as for the more involved Laplacians on bundles or with magnetic or electric potential. Here, both the case of Laplacians on manifolds [9, 38, 45] and on graphs [5, 11, 12] have been studied. In fact, recent years have seen quite a few articles (e.g. [4, 29, 30]) dealing with uniqueness questions for extensions of Laplacians with magnetic potential or Laplacians on bundles over graphs.

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In this paper we present a new approach to studying equality of Dirichlet and Neumann Laplacian on bundles or with magnetic potential. Our approach even deals with uniqueness questions in a substantially more general context. The key element of our approach is to consider the question of uniqueness within the framework of dominating semigroups. Our main abstract theorem (Theorem 2.2) can be seen as stability of uniqueness under a domination condition. To the best of our knowledge it is the very first result of its kind.

As a corollary we obtain that, roughly speaking, Dirichlet and Neumann Laplacian on a bundle agree whenever the corresponding Laplacians on the underlying space agree (Corollaries 2.3 and 2.5). This type of result can then be applied to various examples. In particular, we recover in a new and direct way all known examples for graphs and domains in Euclidean space and provide new examples which were not treated earlier.

In order to establish our results on uniqueness we first have to extend the existing theory of dominating semigroups as developed by Kato [16], Simon [43] and Hess, Schrader, Uhlenbrock [14]. This extension may be of interest in its own right. As discussed above, our approach is concentrated on comparisons of forms on different Hilbert spaces. To achieve this, we give a (partial) generalization of the characterization of domination of semigroups in terms of forms by Ouhabaz [34] and Manavi, Vogt, Voigt [31] in the more abstract setup of [14]. On the technical level this is rather demanding as the actual considerations of [31, 34] are based on pointwise arguments which are not available anymore in the setup of [14]. This treatment might be of special interest for future applications to operators on manifolds as it naturally includes forms on sections of non-trivial vector bundles and more generally on direct integrals of Hilbert spaces.

In retrospect it is not completely surprising that domination of semigroups plays a role in investigation of uniqueness issues. Indeed, as shown by Simon [43], domination of semigroups is equivalent to the validity of a variant of Kato's inequality and this inequality is a key element in all previous proofs of essential self-adjointness of Laplacians with magnetic potential. However, even in those cases where this was shown earlier the actual line of reasoning is quite different from ours: It proceeds by treating the magnetic situation by mimicking the proof given for the Laplacian and invoking Kato's inequality. In this sense, our paper provides the first conceptual connection between uniqueness of extensions and the theory of dominating semigroups. Note, however, that our result does not deal with essential self-adjointness but rather with a somewhat weaker statement that can be thought of as a form of Markov uniqueness (see discussion above). In the final analysis the reason that our methods do not give stability of essential self-adjointness may come from the fact that domination of semigroups concerns the order structure and can therefore only be expected to be of use in connection with extensions respecting some form of order such as extensions given Markov semigroups.

The article is organized as follows: In Section 1 we review the basics of ordered Hilbert spaces and domination of operators and present an abstract version of the characterization of domination of semigroups in terms of the associated forms (Theorem 1.33). In Section 2 we prove the main theorem of this article (Theorem 2.2), a criterion for form uniqueness in terms of domination. In Section 3 we discuss the above mentioned applications, namely Schrödinger forms on vector bundles over graphs, Section 3.1, and Schrödinger forms on domains in Euclidean space, Section 3.2.

The article has its origin in the master's thesis of one of the authors (M. W.).

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1. POSITIVITY IN HILBERT SPACES AND DOMINATION

Domination of operators is a way to compare two operators. It is a generalization of the first Beurling-Deny criterion, which compares an operator to itself. In connection with questions of essential

self-adjointness, it probably occurred first in the form of Kato's inequality comparing Schrödinger operators with and without magnetic field (cf. [16]). Abstractly, it was first defined for operators on L^2 -spaces (cf. [43, 44]) and later generalized by Ouhabaz to the case of one operator acting on a vector valued L^2 -space (cf. [34]). In the framework presented here, which follows [14] and [2], it can be applied to two operators on arbitrary Hilbert spaces as long as there is a so-called symmetrization map between them. Domination will be the main tool in this article to transfer form uniqueness of one operator to form uniqueness of another, dominated operator. This is highly relevant as often the question of uniqueness of form extensions is easier or already proven for the dominating form. Applications include magnetic Schrödinger operators on domains and graphs (cf. Chapter 3).

The main objective of this section will be to give a characterization of domination of semigroups in terms of the associated forms (Theorem 1.33) and to derive a corollary on automatic positivity of dominating forms (Corollary 1.36). In order to do so we will first introduce necessary background on positive forms and on domination of operators in the next two subsections and then turn to the mentioned results in the last subsection.

1.1. Positive cones and positive forms. In this section we collect some basics about order structures in Hilbert spaces induced by positive cone. All the material here is certainly well-known—see [32] for a (very short) introduction. To make this work self-contained, we included proofs of the elementary facts.

As it is convenient for order theory, we only deal with vector spaces over \mathbb{R} in this section.

Definition 1.1 (Positive Cone): *Let \mathcal{K} be a Hilbert space. A closed, non-empty subset \mathcal{K}^+ of \mathcal{K} is called positive cone if*

- (P1) $\mathcal{K}^+ + \mathcal{K}^+ \subset \mathcal{K}^+$,
- (P2) $\alpha\mathcal{K}^+ \subset \mathcal{K}^+$ for all $\alpha \geq 0$,
- (P3) $\langle \mathcal{K}^+, \mathcal{K}^+ \rangle \geq 0$.

For $g_1, g_2 \in \mathcal{K}$ we write

$$g_1 \leq g_2$$

if $g_2 - g_1 \in \mathcal{K}^+$. The positive cone \mathcal{K}^+ is said to be self-dual if

$$\mathcal{K}^+ = \{g \in \mathcal{K} \mid \langle g, h \rangle \geq 0 \text{ for all } h \in \mathcal{K}^+\}.$$

The positive cone \mathcal{K}^+ is called an isotone projection cone if the projection $P_{\mathcal{K}^+}$ onto \mathcal{K}^+ is monotone increasing with respect to \leq , that is, $g \leq h$ implies $P_{\mathcal{K}^+}(g) \leq P_{\mathcal{K}^+}(h)$ for $g, h \in \mathcal{K}$.

Remark: • If $\mathcal{K}^+ \subset \mathcal{K}$ is a positive cone, the dual cone is defined as

$$(\mathcal{K}^+)^{\circ} = \{g \in \mathcal{K} \mid \langle g, h \rangle \geq 0 \text{ for all } h \in \mathcal{K}^+\}.$$

Hence a cone is self-dual if and only if it coincides with its dual. Sometimes the dual cone is also called polar cone (and self-dual cones are called self-polar), but the usual convention seems to be that the polar cone of \mathcal{K}^+ is $-(\mathcal{K}^+)^{\circ}$.

- For self-dual cones it is redundant to assume that they are closed. Indeed,

$$\mathcal{K}^+ = \bigcap_{h \in \mathcal{K}^+} \{g \in \mathcal{K} \mid \langle g, h \rangle \geq 0\}$$

is closed as the intersection of inverse images of a closed set under continuous functions.

In general this is no longer true, as the following example shows: Let $\mathcal{K} = \ell^2$ and $\mathcal{K}^+ = \{f \in \ell^2 \mid f \geq 0, \text{ supp } f \text{ finite}\}$. Then \mathcal{K}^+ satisfies (P1) – (P3), but is obviously not closed.

- Some authors do not assume property (P3) in the definition of positive cones. However, in the case of self-dual cones we are mainly interested in it is automatically satisfied. Our terminology is in accordance with that of [2] and [14], our main sources for Section 1.3.
- The projection P_C onto a closed, convex subset C of a Hilbert space H maps $x \in H$ to the unique element $P_C(x) \in C$ that satisfies $\|P_C(x) - x\| = d(x, C)$. It is characterized as the unique $z \in C$ that satisfies $\langle x - z, y - z \rangle \leq 0$ for all $y \in C$.

In most cases we will be interested in the following example.

Example 1.2: Let (X, \mathcal{B}, m) be a measure space. Then

$$L^2(X, m)^+ = \{f \in L^2(X, m) \mid f \geq 0 \text{ } m\text{-almost everywhere}\}$$

is a self-dual isotone projection cone in $L^2(X, m)$.

Remark: It can be shown (cf. [36], Corollary II.4) that all self-dual isotone projection cones arise in this way. Actually, the result of [36] is even a little stronger: If $\mathcal{K}^+ \subset \mathcal{K}$ is a self-dual cone that induces a lattice ordering on \mathcal{K} , there is a compact space X , a regular finite Borel measure μ on X , and a unitary $U: \mathcal{K} \rightarrow L^2(X, m)$ such that $U\mathcal{K}^+ = L^2(X, m)^+$. (That self-dual isotone projection cones indeed satisfy the assumption of inducing a lattice ordering is the content of Proposition 1.6.)

An important tool when dealing with positive cones is Moreau's Theorem (cf. [27]). It shows that in Hilbert spaces with a positive cone there is an abstract analog of the decomposition of a function into positive and negative part. In the original form it deals with two mutually polar cones, but we will need only the version for a self-dual cone as stated below.

Theorem 1.3 (Moreau): *Let \mathcal{K} be a Hilbert space and $\mathcal{K}^+ \subset \mathcal{K}$ a self-dual positive cone. For $g, h_1, h_2 \in \mathcal{K}$ the following statements are equivalent:*

- (i) $g = h_1 - h_2$, $h_1, h_2 \in \mathcal{K}^+$, $\langle h_1, h_2 \rangle = 0$
- (ii) $h_1 = P_{\mathcal{K}^+}(g)$, $h_2 = P_{\mathcal{K}^+}(-g)$.

Note that under the assumptions of the theorem we have

$$P_{\mathcal{K}^+}(-g) = h_2 = -(g - P_{\mathcal{K}^+}(g))$$

for all $g \in \mathcal{K}$.

Definition 1.4 (Riesz space): *A vector space E with a partial order \leq is called ordered vector space if for all $f, g, h \in E$, $\alpha \geq 0$, the following properties hold:*

- (R1) $f \leq g$ implies $f + h \leq g + h$,
- (R2) $f \leq g$ implies $\alpha f \leq \alpha g$.

If additionally

- (R3) $\{f, g\}$ has a least upper bound, $f \vee g$,

for all $f, g \in E$, then E is called a Riesz space. A subspace F of a Riesz space E is called sublattice if $f, g \in F$ implies $f \vee g \in F$.

If E is an ordered vector space, then a least upper bound can easily be seen to be necessarily unique (if it exists). Clearly, for any f, g in a Riesz space a greatest lower bound exists and is unique. It will be denoted by $f \wedge g$ (if it exists).

Definition 1.5 (Positive and negative part): *Let E be an ordered vector space. For $f \in E$ the positive and negative part are defined as $f^\pm = (\pm f) \vee 0$ if they exist. In this case the absolute value is defined as $|f| = f^+ + f^-$.*

The connection between Riesz spaces and positive cones in Hilbert spaces is given by the following result by Isac and Németh (cf. [15], Proposition 3).

Proposition 1.6: *Let \mathcal{K} be a Hilbert space and $\mathcal{K}^+ \subset \mathcal{K}$ a self-dual isotone projection cone. Then (\mathcal{K}, \leq) is a Riesz space.*

The original proof is quite complicated. However, in our setting, which is a little more restrictive than that in [15], it is a consequence of the next lemma combined with the (easily established) fact that an ordered vector space E is a Riesz space, if f^+ exists for all $f \in E$.

Lemma 1.7: *Let \mathcal{K} be a Hilbert space, $\mathcal{K}^+ \subset \mathcal{K}$ a self-dual isotone projection cone, and $g \in \mathcal{K}$. Then $P_{\mathcal{K}^+}(g)$ is a least upper bound of $\{0, g\}$.*

Proof. Let $g \in \mathcal{K}$. By Moreau's Theorem 1.3 we have

$$g = P_{\mathcal{K}^+}(g) - P_{\mathcal{K}^+}(-g) \leq P_{\mathcal{K}^+}(g).$$

Hence, $P_{\mathcal{K}^+}(g)$ is an upper bound for $\{0, g\}$. Now let h be an upper bound for $\{0, g\}$. By isotonicity we have $P_{\mathcal{K}^+}(g) \leq P_{\mathcal{K}^+}(h) = h$. Thus, $P_{\mathcal{K}^+}(g)$ is the least upper bound of $\{0, g\}$. \square

Lemma 1.8: *Let \mathcal{K} be a Hilbert space and $\mathcal{K}^+ \subset \mathcal{K}$ a self-dual isotone projection cone. Then $\|\cdot\|: \mathcal{K}^+ \rightarrow [0, \infty)$ is monotone increasing and $\|g\| = \||g|\|$ for all $g \in \mathcal{K}$.*

Proof. Let $g, h \in \mathcal{K}^+$ such that $g \leq h$. Then we have

$$0 \leq \langle h - g, g \rangle = \langle h, g \rangle - \|g\|^2 \leq \|h\|\|g\| - \|g\|^2,$$

hence $\|g\| \leq \|h\|$. Moreover,

$$\|g\|^2 = \langle g^+ - g^-, g^+ - g^- \rangle = \langle g^+ + g^-, g^+ + g^- \rangle = \||g|\|^2$$

since $\langle g^+, g^- \rangle = 0$ by Moreau's Theorem 1.3. \square

Having studied some of the basic properties, we will now turn to forms on ordered Hilbert spaces that are compatible with the order structure.

Definition 1.9 (Positive form): *Let \mathcal{K} be a Hilbert space and $\mathcal{K}^+ \subset \mathcal{K}$ a positive cone. A form $(\mathfrak{b}, D(\mathfrak{b}))$ in \mathcal{K} is called positive if $P_{\mathcal{K}^+}D(\mathfrak{b}) \subset D(\mathfrak{b})$ and*

$$\mathfrak{b}(P_{\mathcal{K}^+}(g), P_{\mathcal{K}^+}(-g)) \leq 0$$

for all $g \in D(\mathfrak{b})$.

By the following result of Ouhabaz (see [34], Thm. 2.1 and Proposition 2.3, and [35], Theorem 3), a form is positive if and only if the associated semigroup preserves the positive cone \mathcal{K}^+ , explaining the term “positive form”. This notion should not be confused with that of forms with lower bound 0, which are also sometimes called “positive”.

Proposition 1.10 (Ouhabaz): *Let \mathcal{H} be a Hilbert space, C a closed, convex subset of \mathcal{H} , P the projection onto C , (Q_t) a semigroup on \mathcal{H} with generator T and q the associated form with lower bound $-\lambda$. Then the following are equivalent:*

- (i) $Q_t C \subset C$ for all $t \geq 0$
- (ii) $\alpha(T + \alpha)^{-1}C \subset C$ for all $\alpha > \lambda$
- (iii) $P(D(q)) \subset D(q)$ and $\operatorname{Re} q(Pu, u - Pu) \geq 0$ for all $u \in D(q)$

As already announced the proposition immediately implies the following corollary.

Corollary 1.11 (Characterization of positive forms): *Let \mathcal{K} be a Hilbert space and $\mathcal{K}^+ \subset \mathcal{K}$ a positive cone. Let $(\mathfrak{b}, D(\mathfrak{b}))$ be a form in \mathcal{K} which is semibounded below and closed and B the associated self-adjoint operator. Then $(\mathfrak{b}, D(\mathfrak{b}))$ is positive if and only if e^{-tB} leaves \mathcal{K}^+ invariant for any $t \geq 0$.*

The next lemma gives another characterization of positive forms. We omit the proof since it is well-known for L^2 -spaces and easily carries over to our more abstract setting.

Lemma 1.12: *Let \mathcal{K} be a Hilbert space, $\mathcal{K}^+ \subset \mathcal{K}$ a self-dual isotone projection cone, and $(\mathfrak{b}, D(\mathfrak{b}))$ a form in \mathcal{K} . Then \mathfrak{b} is positive if and only if $|D(\mathfrak{b})| \subset D(\mathfrak{b})$ and $\mathfrak{b}(|g|) \leq \mathfrak{b}(g)$ for all $g \in D(\mathfrak{b})$.*

Some basic properties of positive forms and their domains are gathered next.

Lemma 1.13: *Let \mathcal{K} be a Hilbert space, $\mathcal{K}^+ \subset \mathcal{K}$ a self-dual isotone projection cone, and $(\mathfrak{b}, D(\mathfrak{b}))$ a positive form in \mathcal{K} with lower bound $-\lambda \in \mathbb{R}$.*

- (a) *The form domain $D(\mathfrak{b})$ is a sublattice of \mathcal{K} .*
- (b) *For any $\alpha \geq \lambda$, the form $\mathfrak{b}_\alpha := \mathfrak{b} + \alpha\langle \cdot, \cdot \rangle$ satisfies*

$$\mathfrak{b}_\alpha(g \wedge h), \mathfrak{b}_\alpha(g \vee h) \leq \mathfrak{b}_\alpha(g) + \mathfrak{b}_\alpha(h)$$

for all $g, h \in D(\mathfrak{b})$.

Proof. Let $g, h \in D(\mathfrak{b})$. By Lemma 1.12 we have

$$g \wedge h = \frac{1}{2}(g + h - |g - h|) \in D(\mathfrak{b})$$

and analogously for $g \vee h$. Hence, $D(\mathfrak{b})$ is a sublattice of \mathcal{K} .

Since $\langle P_{\mathcal{K}^+}(g), P_{\mathcal{K}^+}(-g) \rangle = 0$ by Moreau's Theorem 1.3, the form \mathfrak{b}_α is positive. Moreover, $\mathfrak{b}_\alpha(u+v) \geq 0$ implies $-2\mathfrak{b}_\alpha(u, v) \leq \mathfrak{b}_\alpha(u) + \mathfrak{b}_\alpha(v)$ for all $u, v \in D(\mathfrak{b})$. With the aid of this inequality, the positivity of \mathfrak{b}_α and the parallelogram identity we obtain

$$\begin{aligned} \mathfrak{b}_\alpha(g \wedge h) &= \frac{1}{4}\mathfrak{b}_\alpha(g + h - |g - h|) \\ &= \frac{1}{4}(\mathfrak{b}_\alpha(g + h) + \mathfrak{b}_\alpha|g - h|) - 2\mathfrak{b}_\alpha(g + h, |g - h|) \\ &\leq \frac{1}{2}(\mathfrak{b}_\alpha(g + h) + \mathfrak{b}_\alpha(|g - h|)) \\ &\leq \frac{1}{2}(\mathfrak{b}_\alpha(g + h) + \mathfrak{b}_\alpha(g - h)) \\ &= \mathfrak{b}_\alpha(g) + \mathfrak{b}_\alpha(h). \end{aligned}$$

The result for $\mathfrak{b}_\alpha(g \vee h)$ follows similarly. \square

Remark (Real versus complex Hilbert spaces): So far we developed a theory developed for real Hilbert spaces only as that completely serves our purposes. To incorporate complex Hilbert spaces, one can proceed as follows:

Every complex Hilbert space \mathcal{K} becomes a real Hilbert space \mathcal{K}_r when equipped with the inner product $\langle \cdot, \cdot \rangle_r = \operatorname{Re} \langle \cdot, \cdot \rangle$ and a positive cone \mathcal{K}^+ in \mathcal{K} is also a positive cone in \mathcal{K}_r . However, self-duality of \mathcal{K}^+ is not preserved, but \mathcal{K}_r decomposes as

$$\mathcal{K}_r = \mathcal{K}^J \oplus i\mathcal{K}^J$$

with $\mathcal{K}^J = \mathcal{K}^+ - \mathcal{K}^+$, and \mathcal{K}^+ is a self-dual cone in \mathcal{K}^J . Therefore, every element $g \in \mathcal{K}^+$ has a unique decomposition as

$$g = g_1 - g_2 + i(g_3 - g_4)$$

with $g_1, \dots, g_4 \in \mathcal{K}^+$ and $\langle g_i, g_j \rangle = 0$ for $i \neq j$. This decomposition yields an anti-unitary involution J via

$$J: \mathcal{K}^J \oplus i\mathcal{K}^J \longrightarrow \mathcal{K}^J \oplus i\mathcal{K}^J, g + ih \mapsto g - ih.$$

Positive forms on \mathcal{K} are real in the sense that $JD(\mathfrak{b}) = D(\mathfrak{b})$ and $\mathfrak{b}(Jg) = \mathfrak{b}(g)$. Thus, there is no loss of generality when dealing exclusively with real Hilbert spaces as we can always restrict to \mathcal{K}^J in the complex case.

1.2. Symmetrization and domination of operators. In this section we introduce the concept of domination of operators. As mentioned already this concept allows one to compare two operators. These operators may act on different Hilbert spaces provided that there is a modulus type map between these Hilbert spaces. Such a map is called symmetrization.

Throughout this section \mathcal{K} denotes a real Hilbert space and \mathcal{H} a Hilbert space, either real or complex.

Definition 1.14 (Symmetrization): *Let $\mathcal{K}^+ \subset \mathcal{K}$ be a positive cone. A map $S: \mathcal{H} \longrightarrow \mathcal{K}^+$ is called absolute mapping if*

$$(S1) \quad |\langle f_1, f_2 \rangle| \leq \langle S(f_1), S(f_2) \rangle \text{ for all } f_1, f_2 \in \mathcal{H} \text{ and equality if } f_1 = f_2.$$

An absolute mapping is called absolute pairing or symmetrization if

$$(S2) \quad \text{For any } g \in \mathcal{K}^+ \text{ and } f_1 \in \mathcal{H} \text{ there is an } f_2 \in \mathcal{H} \text{ such that } g = S(f_2) \text{ and}$$

$$\langle f_1, f_2 \rangle = \langle S(f_1), S(f_2) \rangle = \langle S(f_1), g \rangle.$$

In this case f_1 and f_2 are called g -paired or simply paired.

Remark: • Note that

$$\langle f, f \rangle = \langle S(f), S(f) \rangle$$

holds for any $f \in \mathcal{K}$, whenever S is an absolute mapping on \mathcal{K} .

- By its very definition any symmetrization is onto.

If one thinks of an absolute mapping S as a form of modulus, then one may think of f_2 appearing in (S2) of the preceding definition as a form of $\text{sgn}(f_1) \cdot g$ (see the examples below for further justification of this point of view). In this sense a symmetrization is a modulus type map together with the possibility of forming a signum.

The following lemma shows that we have already encountered a natural example of an absolute mapping in the last section.

Lemma 1.15: *Let $\mathcal{K}^+ \subset \mathcal{K}$ be a self-dual, isotone projection cone. Then $|\cdot|: \mathcal{K} \rightarrow \mathcal{K}^+$ is an absolute mapping.*

Proof. Let $g, h \in \mathcal{K}$. Then we have

$$\begin{aligned} |\langle g, h \rangle| &= |\langle g^+ - g^-, h^+ - h^- \rangle| \\ &= |\langle g^+, h^+ \rangle - \langle g^-, h^+ \rangle - \langle g^+, h^- \rangle + \langle g^-, h^- \rangle| \\ &\leq \langle g^+, h^+ \rangle + \langle g^-, h^+ \rangle + \langle g^+, h^- \rangle + \langle g^-, h^- \rangle \\ &= \langle g^+ + g^-, h^+ + h^- \rangle \\ &= \langle |g|, |h| \rangle. \end{aligned}$$

Equality in the case $g = h$ was already shown in Lemma 1.8. \square

In a spatial setting the modulus together with a signum function provide symmetrizations as discussed in the next examples.

Example 1.16: The norm $\|\cdot\|: \mathcal{H} \rightarrow [0, \infty)$ is a symmetrization. For $\lambda > 0$ and $f_1 \in \mathcal{H}$ an element $f_2 \in \mathcal{H}$ such that f_1 and f_2 are λ -paired is given by $f_2 = \lambda \frac{f_1}{\|f_1\|}$ if $f_1 \neq 0$, and by $f_2 = \lambda \xi$ for any $\xi \in \mathcal{H}$ with $\|\xi\| = 1$ if $f_1 = 0$.

Example 1.17: Let (X, \mathcal{B}, m) be a measure space and H a Hilbert space. Denote the norm on H by $|\cdot|$. Then $S: L^2(X, m; H) \rightarrow L^2(X, m)$, $(Sf)(x) = |f(x)|$ is a symmetrization. Property (S1) is obvious. For (S2) choose $f_2 = g \text{sgn}_\xi f_1$, where $\text{sgn}_\xi f_1$ is defined by

$$\text{sgn}_\xi f_1(x) = \begin{cases} \frac{f_1(x)}{|f_1(x)|} & : f_1(x) \neq 0 \\ \xi & : f_1(x) = 0 \end{cases}$$

for some $\xi \in H$ with $|\xi| = 1$. Then f_1 and f_2 are g -paired.

Example 1.18: Let X be a topological space, m a Borel measure on X and E a Hermitian vector bundle over X , that is, a vector bundle with an inner product on the fibers that varies continuously with the base point (see [28], where the term *Euclidean vector bundle* is used). Denote by $L^2(X, m; E)$ the space of L^2 -sections in E . Then

$$S: L^2(X, m; E) \rightarrow L^2(X, m)^+, (Sf)(x) = |f(x)|_x$$

is a symmetrization by the same arguments as in the previous example.

In the following lemma we collect some basic properties of the modulus that carry over to abstract symmetrizations. See [14], Proposition 2.6, for a proof.

Lemma 1.19: *Let $\mathcal{K}^+ \subset \mathcal{K}$ be a positive cone, and $S: \mathcal{H} \rightarrow \mathcal{K}^+$ a symmetrization.*

(a) *The triangle inequality*

$$\langle S(f_1 + f_2), g \rangle \leq \langle S(f_1) + S(f_2), g \rangle$$

holds for all $f_1, f_2 \in \mathcal{H}$ and $g \in \mathcal{K}^+$.

(b) The symmetrization S is positive homogeneous, i.e.

$$S(\alpha f) = |\alpha|S(f)$$

holds for all $f \in \mathcal{H}$ and $\alpha \in \mathbb{C}$.

(c) The symmetrization S is positive definite: For all $f \in \mathcal{H}$, $S(f) = 0$ if and only if $f = 0$.

(d) The map S is Lipschitz continuous.

To a certain extend symmetrization is compatible with taking differences (as it the case for the usual modulus).

Lemma 1.20: *Let $\mathcal{K}^+ \subset \mathcal{K}$ be a positive cone and $S: \mathcal{H} \rightarrow \mathcal{K}^+$ a symmetrization. If $f_1, f_2 \in \mathcal{H}$ satisfy $S(f_2) \leq S(f_1)$ and are $S(f_2)$ -paired, then $S(f_1 - f_2) = S(f_1) - S(f_2)$ and $f_1 - f_2, f_2$ are $S(f_2)$ -paired.*

Proof. By the triangle inequality, $S(f_1 - f_2) \geq S(f_1) - S(f_2)$ holds. Moreover we have

$$\begin{aligned} \|S(f_1 - f_2)\|^2 &= \|f_1 - f_2\|^2 \\ &= \|f_1\|^2 + \|f_2\|^2 - 2\langle f_1, f_2 \rangle \\ &= \|S(f_1)\|^2 + \|S(f_2)\|^2 - 2\langle S(f_1), S(f_2) \rangle \\ &= \|S(f_1) - S(f_2)\|^2. \end{aligned}$$

Let $g, g' \in \mathcal{K}^+$ such that $g \leq g'$ and $\|g\| = \|g'\|$. Then we have

$$\|g - g'\|^2 = \|g\|^2 + \|g'\|^2 - 2\langle g, g' \rangle = 2\|g\|^2 - 2\langle g, g + g' - g \rangle = -2\langle g, g' - g \rangle \leq 0.$$

Hence $g = g'$. Applying this result to $g = S(f_1) - S(f_2)$, $g' = S(f_1 - f_2)$, we obtain the desired equality for $S(f_2 - f_1)$. Moreover,

$$\langle f_2 - f_1, f_2 \rangle = \langle S(f_2) - S(f_1), S(f_2) \rangle = \langle S(f_1 - f_2), S(f_2) \rangle,$$

hence $f_1 - f_2$ and f_2 are paired. \square

The next lemma will serve as a characterization of the central concept of this section, namely domination of operators. A proof is given in [14], Proposition 2.7.

Lemma 1.21: *Let $\mathcal{K}^+ \subset \mathcal{K}$ be a positive cone and $S: \mathcal{H} \rightarrow \mathcal{K}^+$ a symmetrization. For bounded operators P (resp. Q) on \mathcal{H} (resp. \mathcal{K}), the following are equivalent:*

- (i) $\langle S(Pf_1), g \rangle \leq \langle QS(f_1), g \rangle$ for all $f_1 \in \mathcal{H}$, $g \in \mathcal{K}^+$
- (ii) $\operatorname{Re}\langle Pf_1, f_2 \rangle \leq \langle QS(f_1), S(f_2) \rangle$ for all $f_1, f_2 \in \mathcal{H}$
- (iii) $|\langle Pf_1, f_2 \rangle| \leq \langle QS(f_1), S(f_2) \rangle$ for all $f_1, f_2 \in \mathcal{H}$

Furthermore, if \mathcal{K}^+ is self-dual, these assertions are equivalent to

- (iv) $S(Pf_1) \leq QS(f_1)$ for all $f_1 \in \mathcal{H}$.

Definition 1.22 (Domination of operators): *If P and Q satisfy one of the equivalent assertions of Lemma 1.21, then P is said to be dominated by Q . Likewise a family of bounded operators $(P_\alpha)_{\alpha \in J}$ is said to be dominated by the family of bounded operators $(Q_\alpha)_{\alpha \in J}$ if P_α is dominated by Q_α for each $\alpha \in J$.*

By its very definition the domination of P by Q can be read as the operator Q entailing properties of the operator P (and this will be our point of view in our main result below). Note, however, that the definition also implies some structural positivity property of Q as $Q(\mathcal{K}^+)$ must be a subset of \mathcal{K}^+ . We will meet this positivity property in various places below.

Example 1.23: Let (X, \mathcal{B}, m) be a measure space and $P: L^2(X, m) \rightarrow L^2(X, m)$ a linear operator that leaves $L^2(X, m)^+$ invariant. Then P is dominated by itself: For all $f \in L^2(X, m)$, we have

$$|Pf| = |P(f^+ - f^-)| \leq |Pf^+| + |Pf^-| = Pf^+ + Pf^- = P|f|.$$

Indeed, also the converse is true: If P is dominated by itself, then $Pf = P|f| \geq |Pf| \geq 0$ for all $f \in L^2(X, m)^+$, hence $PL^2(X, m)^+ \subset L^2(X, m)^+$.

We will give more interesting examples (in particular such that have different operators P and Q) once we have a characterization of domination of semigroups in terms of the associated forms at hand. But before we turn to this characterization, we present some basic algebraic properties of domination. A proof is given in [2], Appendix A, Lemma 14.

Lemma 1.24: *Let $\mathcal{K}^+ \subset \mathcal{K}$ be a positive cone, $S: \mathcal{H} \rightarrow \mathcal{K}^+$ a symmetrization, and P_1, P_2 (resp. Q_1, Q_2) bounded self-adjoint operators on \mathcal{H} (resp. \mathcal{K}).*

- (a) *Let $\alpha_1, \alpha_2 \in \mathbb{C}$. If P_i is dominated by Q_i , $i \in \{1, 2\}$, then $\alpha_1 P_1 + \alpha_2 P_2$ is dominated by $|\alpha_1| Q_1 + |\alpha_2| Q_2$.*
- (b) *If \mathcal{K}^+ is self-dual and P_1 is dominated by Q_1 , then Q_1 preserves the cone \mathcal{K}^+ .*
- (c) *If P_i is dominated by Q_i , $i \in \{1, 2\}$, and Q_1 preserves \mathcal{K}^+ , then $P_1 P_2$ is dominated by $Q_1 Q_2$.*

Part (b) of the previous Lemma can be combined with Corollary 1.11 to give the following result.

Corollary 1.25 (Domination implies positivity): *Let $\mathcal{K}^+ \subset \mathcal{K}$ be a self-dual positive cone and $S: \mathcal{H} \rightarrow \mathcal{K}^+$ a symmetrization. If the form $(\mathbf{b}, D(\mathbf{b}))$ in \mathcal{K} is semibounded below and closed with associated self-adjoint operator B such that the semigroup e^{-tB} , $t \geq 0$, dominates a family A_t , $t \geq 0$, of bounded self-adjoint operators in \mathcal{K} , then the form $(\mathbf{b}, D(\mathbf{b}))$ is positive.*

1.3. Characterizing domination of operators via forms. In this section we characterize domination of semigroups via a domination property of forms. Roughly speaking, we do so by combining the framework provided in [14] with the methods provided in [31, 34]. More specifically, we follow the strategy of [31, 34] and first provide below a characterization of domination of semigroups by the invariance of a certain convex set and then relate this invariance to a positivity property of the form by means of the result of [34], Proposition 1.10.

The next proposition is the main tool in the characterization of domination of semigroups in terms of the associated forms. It is an abstract version of Lemma 3.2 in [31]. The proof given there relies on pointwise considerations that are not applicable in our setting. Instead we will have to rely on the abstract properties of absolute mappings and isotone projection cones. This makes the proof technically demanding (and somewhat lengthy as well).

Proposition 1.26 (Domination via invariance of C): *Let $\mathcal{K}^+ \subset \mathcal{K}$ be a self-dual positive cone, $S: \mathcal{H} \rightarrow \mathcal{K}^+$ an absolute mapping, and (P_t) (resp. (Q_t)) a semigroup on \mathcal{H} (resp. \mathcal{K}). Define the semigroup (W_t) on $\mathcal{H} \oplus \mathcal{K}$ via*

$$W_t(f, g) = (P_t f, Q_t g)$$

for $t \geq 0$, $(f, g) \in \mathcal{H} \oplus \mathcal{K}$. Define the set

$$C = \{(u, v) \in \mathcal{H} \oplus \mathcal{K} \mid S(u) \leq v\}.$$

- (a) *The set C is a closed, convex subset of $\mathcal{H} \oplus \mathcal{K}$.*
- (b) *The semigroup (P_t) is dominated by (Q_t) if and only if C is invariant under (W_t) .*
- (c) *Let $g \in \mathcal{K}^+$ and $f_1 \in \mathcal{H}$ with $g \leq S(f_1)$. Whenever there is an $f_2 \in \mathcal{H}$ such that f_1, f_2 are g -paired, the projection P_C onto C satisfies*

$$P_C(f_1, g) = \frac{1}{2}(f_1 + f_2, S(f_1) + g).$$

- (d) *If \mathcal{K}^+ is a self-dual isotone projection cone, the projection P_C onto C satisfies*

$$P_C(f_1, g) = \frac{1}{2}(f_2, (S(f_1) \vee g + g)^+),$$

for $f_1 \in \mathcal{H}, g \in \mathcal{K}$ whenever there is an $f_2 \in \mathcal{H}$ such that f_1, f_2 are $(S(f_1) \wedge g + S(f_1))^+$ -paired.

Remark: Of course, the existence of an element $f_2 \in \mathcal{H}$ as in (c), (d) is automatically guaranteed if S is actually a symmetrization. However, for the following corollary we need the above proposition when S is the absolute value on \mathcal{K} , which is not necessarily a symmetrization.

Proof. (a) Since S is positive homogeneous and satisfies the triangle inequality (Lemma 1.19), it is clear that C is convex. By Lemma 1.19, S is continuous. Thus, C is closed as the preimages of \mathcal{K}^+ under the continuous map

$$\varphi: \mathcal{H} \oplus \mathcal{K} \longrightarrow \mathcal{K}, (u, v) \mapsto v - S(u).$$

(b) First assume that C is invariant under (W_t) . Let $f \in \mathcal{H}$ and $t \geq 0$. Then we have $(f, S(f)) \in C$, hence

$$(P_t f, Q_t S(f)) = W_t(f, S(f)) \in C,$$

that is, $S(P_t f) \leq Q_t S(f)$. Thus, (P_t) is dominated by (Q_t) . Conversely assume (P_t) is dominated by (Q_t) . Let $(u, v) \in C$. Since \mathcal{K}^+ is self-dual, the semigroup (Q_t) leaves \mathcal{K}^+ invariant by Lemma 1.24. Thus, $S(u) \leq v$ implies $S(P_t u) \leq Q_t S(u) \leq Q_t v$. Hence, $W_t(u, v) = (P_t u, Q_t v) \in C$.

(c) Let $f_1 \in \mathcal{H}$, $g \in \mathcal{K}^+$ such that $g \leq S(f_1)$ and assume there is an $f_2 \in \mathcal{H}$ such that f_1, f_2 are g -paired. Define

$$P(f_1, g) = \frac{1}{2}(f_1 + f_2, S(f_1) + g).$$

The projection (\hat{f}_1, \hat{g}) of (f_1, g) onto C is characterized as the unique element in C satisfying

$$\operatorname{Re}\langle (f_1, g) - (\hat{f}_1, \hat{g}), (u, v) - (\hat{f}_1, \hat{g}) \rangle \leq 0$$

for all $(u, v) \in C$. We will show that $P(f_1, g) = (\hat{f}_1, \hat{g})$.

Since $S((f_1 + f_2)) \leq S(f_1) + S(f_2) = S(f_1) + g$, $P(f_1, g)$ is an element of C . For all $(u, v) \in C$ we have

$$\begin{aligned} & \operatorname{Re}\langle (f_1, g) - P(f_1, g), (u, v) - P(f_1, g) \rangle \\ &= \frac{1}{4} \operatorname{Re}\langle (f_1 - f_2, g - S(f_1)), (2u - f_1 - f_2, 2v - S(f_1) - g) \rangle \\ &= \frac{1}{4} \operatorname{Re}(\langle f_1 - f_2, 2u \rangle - \|f_1\|^2 + \|f_2\|^2 + 2\langle g - S(f_1), v \rangle - \|g\|^2 + \|S(f_1)\|^2) \\ &= \frac{1}{2} \operatorname{Re}(\langle f_1 - f_2, u \rangle + \langle g - S(f_1), v \rangle) \end{aligned}$$

Since $S(f_2) = g \leq S(f_1)$, we have $S(f_1 - f_2) = S(f_1) - S(f_2)$ by Lemma 1.20 and therefore

$$|\langle f_1 - f_2, u \rangle| \leq \langle S(f_1 - f_2), S(u) \rangle \leq \langle S(f_1) - S(f_2), v \rangle.$$

This implies

$$\begin{aligned} \frac{1}{2} \operatorname{Re}(\langle f_1 - f_2, u \rangle + \langle g - S(f_1), v \rangle) &\leq \frac{1}{2} \operatorname{Re}(\langle g - S(f_1), v \rangle + |\langle f_1 - f_2, u \rangle|) \\ &\leq \frac{1}{2} \operatorname{Re}(\langle g - S(f_1), v \rangle + \langle S(f_1) - g, v \rangle) \\ &= 0. \end{aligned}$$

Hence, $P(f_1, g)$ is the projection of (f_1, g) on C .

(d) Let $f_1 \in \mathcal{H}$, $g \in \mathcal{K}$ and $f_2 \in \mathcal{H}$ such that f_1, f_2 are $(S(f_1) \wedge g + S(f_1))^+$ -paired. Define

$$P(f_1, g) = \frac{1}{2}(f_2, (S(f_1) \vee g + g)^+).$$

As in (c), we will show $P = P_C$ via the characterization of P_C given above.

Since \mathcal{K}^+ is an isotone projection cone, $S(f_1) \wedge g + S(f_1) \leq g + S(f_1) \vee g$ implies $(S(f_1) \wedge g + S(f_1))^+ \leq (S(f_1) \vee g + g)^+$, hence $P(f_1, g) \in C$.

So we have to show that

$$\operatorname{Re}\langle (f_1, g) - P(f_1, g), (u, v) - P(f_1, g) \rangle \leq 0$$

for all $(u, v) \in C$.

We will evaluate the terms

$$I = \operatorname{Re}\langle (f_1, g) - P(f_1, g), (u, v) \rangle \text{ and } J = \langle (f_1, g) - P(f_1, g), -P(f_1, g) \rangle$$

separately. Using $|\langle f, \tilde{f} \rangle| \leq \langle S(f), S(\tilde{f}) \rangle$ for $f, \tilde{f} \in \mathcal{H}$, and $S(u) \leq v$, we obtain

$$\begin{aligned} I &= \operatorname{Re} \langle f_1 - \frac{1}{2}f_2, u \rangle + \langle g - \frac{1}{2}(S(f_1) \vee g + g)^+, v \rangle \\ &\leq \langle S(f_1 - \frac{1}{2}f_2), S(u) \rangle + \langle g - \frac{1}{2}(S(f_1) \vee g + g)^+, v \rangle \\ &\leq \langle S(f_1 - \frac{1}{2}f_2) + g - \frac{1}{2}(S(f_1) \vee g + g)^+, v \rangle. \end{aligned}$$

Lemma 1.20 implies

$$S(f_1 - \frac{1}{2}f_2) = S(f_1) - \frac{1}{2}S(f_2) = S(f_1) - \frac{1}{2}(S(f_1) \wedge g + S(f_1))^+.$$

Thus,

$$\begin{aligned} I &\leq \langle S(f_1) - \frac{1}{2}(S(f_1) \wedge g + S(f_1))^+ + g - \frac{1}{2}(S(f_1) \vee g + g)^+, v \rangle \\ &\leq \langle S(f_1) + g - \frac{1}{2}(S(f_1) \wedge g + S(f_1)) - \frac{1}{2}(S(f_1) \vee g + g), v \rangle \\ &= \langle S(f_1) + g - \frac{1}{2}(S(f_1) + g + S(f_1) + g), v \rangle \\ &= 0, \end{aligned}$$

where we used $h \leq h^+$ and $h \wedge \tilde{h} + h \vee \tilde{h} = h + \tilde{h}$ for $h, \tilde{h} \in \mathcal{K}$.

Next let us turn to J :

$$\begin{aligned} J &= \langle f_1 - \frac{1}{2}f_2, -\frac{1}{2}f_2 \rangle + \langle g - \frac{1}{2}(S(f_1) \vee g + g)^+, -\frac{1}{2}(S(f_1) \vee g + g)^+ \rangle \\ &= -\frac{1}{2}\langle S(f_1), S(f_2) \rangle + \frac{1}{4}\langle S(f_2), S(f_2) \rangle - \frac{1}{2}\langle g, (S(f_1) \vee g + g)^+ \rangle \\ &\quad + \frac{1}{4}\langle (S(f_1) \vee g + g)^+, (S(f_1) \vee g + g)^+ \rangle \\ &= -\frac{1}{2}\langle S(f_1), (S(f_1) \wedge g + S(f_1))^+ \rangle + \frac{1}{4}\|(S(f_1) \wedge g + S(f_1))^+\|^2 \\ &\quad - \frac{1}{2}\langle g, (S(f_1) \vee g + g)^+ \rangle + \frac{1}{4}\|(S(f_1) \vee g + g)^+\|^2. \end{aligned}$$

Since positive and negative part are orthogonal to each other, we can write

$$\|(S(f_1) \wedge g + S(f_1))^+\|^2 = \langle S(f_1) \wedge g + S(f_1), (S(f_1) \wedge g + S(f_1))^+ \rangle$$

and likewise for $\|(S(f_1) \vee g + g)^+\|^2$.

Using once again $h \wedge \tilde{h} + h \vee \tilde{h} = h + \tilde{h}$ for $h, \tilde{h} \in \mathcal{K}$, it follows that

$$\begin{aligned} 4J &= \langle S(f_1) \wedge g - S(f_1), (S(f_1) \wedge g + S(f_1))^+ \rangle \\ &\quad + \langle S(f_1) \vee g - g, (S(f_1) \vee g + g)^+ \rangle \\ &= \langle g - S(f_1) \vee g, (S(f_1) \wedge g + S(f_1))^+ \rangle + \langle S(f_1) \vee g - g, (S(f_1) \vee g + g)^+ \rangle \\ &= \langle S(f_1) \vee g - g, (S(f_1) \vee g + g)^+ - (S(f_1) \wedge g + S(f_1))^+ \rangle. \end{aligned}$$

We analyze the factors of the inner product separately.

As for the first factor, isotonicity implies that

$$(S(f_1) \vee g + g)^+ - (S(f_1) \wedge g + S(f_1))^+ \geq (S(f_1) + g)^+ - (g + S(f_1))^+ = 0$$

and

$$(S(f_1) \vee g + g)^+ - (S(f_1) \wedge g + S(f_1))^+ \geq (2g)^+ - 2S(f_1) \geq 2(g - S(f_1)),$$

hence

$$(S(f_1) \vee g + g)^+ - (S(f_1) \wedge g + S(f_1))^+ \geq 2(g - S(f_1))^+.$$

Moreover

$$\begin{aligned} (S(f_1) \vee g + g) - (S(f_1) \wedge g + S(f_1)) &= g - S(f_1) + S(f_1) \vee g - S(f_1) \wedge g \\ &= g - S(f_1) + |g - S(f_1)| \\ &= 2(g - S(f_1))^+. \end{aligned}$$

Let $h, \tilde{h} \in \mathcal{K}$ such that $\tilde{h} - h \geq 0$, $\tilde{h}^+ - h^+ \geq \tilde{h} - h$. Then

$$\tilde{h}^- - h^- = (\tilde{h}^+ - h^+) - (\tilde{h} - h) \geq 0.$$

On the other hand, $\tilde{h} \geq h$ implies by isotonicity $\tilde{h}^- \leq h^-$. Combining both inequalities yields $\tilde{h}^- = h^-$ and consequently $\tilde{h}^+ - h^+ = \tilde{h} - h$.

Applied to $\tilde{h} = S(f_1) \vee g + g$ and $h = S(f_1) \wedge g + S(f_1)$ this means that

$$(S(f_1) \vee g + g)^+ - (S(f_1) \wedge g + S(f_1))^+ = 2(g - S(f_1))^+.$$

For the other factor in the inner product expression for $4J$ we have:

$$\begin{aligned} S(f_1) \vee g - g &= \frac{1}{2}(S(f_1) + g + |S(f_1) - g|) - g \\ &= \frac{1}{2}(S(f_1) - g + |S(f_1) - g|) \\ &= (S(f_1) - g)^+ \\ &= (g - S(f_1))^- . \end{aligned}$$

Thus,

$$\begin{aligned} 4J &= \langle S(f_1) \vee g - g, (S(f_1) \vee g + g)^+ - (S(f_1) \wedge g + S(f_1))^+ \rangle \\ &= 2\langle (g - S(f_1))^-, (g - S(f_1))^+ \rangle \\ &= 0. \end{aligned}$$

Combining the results for I and J we finally obtain the desired result:

$$\operatorname{Re}\langle (f_1, g) - P(f_1), (u, v) - P(f_1, g) \rangle = I + \operatorname{Re} J \leq 0. \quad \square$$

Corollary 1.27: *Let $\mathcal{K}^+ \subset \mathcal{K}$ be a self-dual positive cone and $S: \mathcal{H} \rightarrow \mathcal{K}^+$ a symmetrization. If $f_1 \in \mathcal{H}$ and $g \in \mathcal{K}^+$ with $g \leq S(f_1)$, then the element $f_2 \in \mathcal{H}$ such that f_1, f_2 are g -paired is unique.*

Proof. This follows from Proposition 1.26 (c) since the projection is unique. \square

Corollary 1.28: *Let $\mathcal{K}^+ \subset \mathcal{K}$ be a self-dual isotone projection cone and \mathbf{b} a positive form in \mathcal{K} . Let*

$$\pi: \mathcal{K}^+ \oplus \mathcal{K} \rightarrow \mathcal{K}^+ \oplus \mathcal{K}^+, (h, g) \mapsto \frac{1}{2}((h \wedge g + h)^+, (h \vee g + g)^+).$$

Then $\pi(D(\mathbf{b}) \oplus D(\mathbf{b})) \subset D(\mathbf{b}) \oplus D(\mathbf{b})$ and

$$(\mathbf{b} \oplus \mathbf{b})(\pi(h, g), (1 - \pi)(h, g)) \geq 0$$

holds for all $h \in D(\mathbf{b})^+, g \in D(\mathbf{b})$.

Proof. Let (P_t) be the semigroup associated with \mathbf{b} . By Proposition 1.10, (P_t) preserves \mathcal{K}^+ and in Example 1.23 it was shown that (P_t) is dominated by itself.

By Proposition 1.26,

$$C = \{(u, v) \in \mathcal{K} \oplus \mathcal{K} \mid |u| \leq v\}$$

is invariant under $(P_t \oplus P_t)$.

Since two positive elements of \mathcal{K} are obviously paired (with respect to $|\cdot|$), we can apply Proposition 1.26 to deduce that the projection onto C satisfies

$$P(h, g) = \frac{1}{2}((h \wedge g + h)^+, (h \vee g + g)^+) = \pi(h, g)$$

for all $g \in \mathcal{K}, h \in \mathcal{K}^+$.

One more application of Proposition 1.10 yields $\pi(D(\mathfrak{b}) \oplus D(\mathfrak{b})) \subset D(\mathfrak{b}) \oplus D(\mathfrak{b})$ and

$$(\mathfrak{b} \oplus \mathfrak{b})(\pi(h, g), (1 - \pi)(h, g)) \geq 0$$

for all $h \in D(\mathfrak{b})^+, g \in D(\mathfrak{b})$. \square

The following two definitions describing the relation of subspaces under an order structure will be used in the characterization of domination of semigroups. What we call an ideal is a slight generalization of the notion that is often also found under the name order ideal (the common use is restricted to the case of the absolute value mapping as described below). The second notion, generalized ideal, was originally coined by Ouhabaz (cf. [34]) under the name ideal, but that collides with the usual terminology in order theory, so we have adopted the usage of [31].

Definition 1.29 (Ideal): *Let $\mathcal{K}^+ \subset \mathcal{K}$ be a positive cone, $S: \mathcal{H} \rightarrow \mathcal{K}^+$ an absolute mapping, and $U, V \subset \mathcal{H}$ subspaces. Then U is called an ideal of V if for $u \in U, v \in V, S(v) \leq S(u)$ implies $v \in U$.*

If $\mathcal{K}^+ \subset \mathcal{K}$ is a self-dual isotone projection cone, ideals in \mathcal{K} are understood with respect to the absolute mapping $|\cdot|: \mathcal{K} \rightarrow \mathcal{K}^+$.

Definition 1.30 (Generalized ideal): *Let $\mathcal{K}^+ \subset \mathcal{K}$ be a positive cone and $S: \mathcal{H} \rightarrow \mathcal{K}^+$ a symmetrization. A subspace $U \subset \mathcal{H}$ is called generalized ideal of the subspace $V \subset \mathcal{K}$ if the following properties hold:*

- (I1) $S(f) \in V$ for all $f \in U$,
- (I2) For all $f_1 \in U$ and $g \in V^+$ such that $g \leq S(f_1)$ there is an $f_2 \in U$ such that f_1, f_2 are g -paired.

This notion is obviously closely related to that of a symmetrization on U . Notice however that contrary to the definition of a symmetrization, we only demand the existence of $f_2 \in U$ if $g \leq S(f_1)$ here, so that S does not necessarily restrict to a symmetrization $U \rightarrow V^+$.

Remark:

- Let U be a generalized ideal of V . If $f_1 \in U, g \in V$ such that $g \leq S(f_1)$, there is only one $f_2 \in \mathcal{H}$ such that f_1, f_2 are g -paired by Corollary 1.27. Then condition (I2) implies that $f_2 \in U$.
- Despite the terminology, not every generalized ideal is an ideal (even in situations when both notions would make sense). As indicated above, this terminology arose due to historical reasons. However, there are some connections between ideals and generalized ideals as discussed for example in [31], Proposition 3.6 and Corollary 3.7.

Example 1.31: Let X be a topological space, m a Borel measure on X and E a Hermitian vector bundle over X . Then the subspace $U \subset L^2(X, m; E)$ is a generalized ideal of the subspace $V \subset L^2(X, m)$ if and only if

- $u \in U$ implies $|u| \in V$,
- $u \in U, v \in V^+, v \leq |u|$ implies $v \operatorname{sgn} u \in U$.

Since $v \leq |u|$ in the second condition, it is irrelevant which value $\operatorname{sgn} u$ has at the zeros of u , and we can stick to the usual convention $\operatorname{sgn} u(x) = 0$ if $u(x) = 0$ instead of taking $\operatorname{sgn}_\epsilon$ from Example 1.17.

Definition 1.32 (Domination of forms): *Let $\mathcal{K}^+ \subset \mathcal{K}$ be a positive cone, $S: \mathcal{H} \rightarrow \mathcal{K}^+$ a symmetrization, and \mathfrak{a} (resp. \mathfrak{b}) a closed form on \mathcal{H} (resp. \mathcal{K}). Then \mathfrak{a} is said to be dominated by \mathfrak{b} if $D(\mathfrak{a})$ is a generalized ideal of $D(\mathfrak{b})$ and*

$$\operatorname{Re} \mathfrak{a}(f_1, f_2) \geq \mathfrak{b}(S(f_1), S(f_2))$$

holds for all $f_1, f_2 \in D(\mathfrak{a})$ that are $S(f_2)$ -paired.

Now we can finally give the characterization of domination of semigroups in terms of the associated forms. For comments on the history of this theorem as well as on the proof, see the remarks below.

Theorem 1.33 (Characterization of domination of semigroups via forms): *Let $\mathcal{K}^+ \subset \mathcal{K}$ be a self-dual positive cone, $S: \mathcal{H} \rightarrow \mathcal{K}^+$ a symmetrization, A (resp. B) a self-adjoint operator on \mathcal{H} (resp. \mathcal{K}) with lower bound $-\lambda$, and \mathfrak{a} (resp. \mathfrak{b}) the associated form. Then the following assertions are equivalent:*

- (i) *The semigroup $(e^{-tA})_{t \geq 0}$ is dominated by $(e^{-tB})_{t \geq 0}$.*
- (ii) *The resolvent $((A + \alpha)^{-1})_{\alpha > \lambda}$ is dominated by $((B + \alpha)^{-1})_{\alpha > \lambda}$.*

Both assertions imply

- (iii) *The form \mathfrak{a} is dominated by \mathfrak{b} .*

If \mathcal{K}^+ is a self-dual isotone projection cone, the assertions (i), (ii) and (iii) are equivalent.

Proof. The equivalence of (i) and (ii) follows quite easily from the correspondence between semigroups and associated resolvents (see [2], Appendix A, Corollary 15).

Next we do some preparatory work for (i) \implies (iii) and (iii) \implies (i) in the isotone case.

Define $C = \{(u, v) \in \mathcal{H} \oplus \mathcal{K} \mid S(u) \leq v\}$ and $W_t = e^{-tA} \oplus e^{-tB}: \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{H} \oplus \mathcal{K}$ for $t \geq 0$.

By Proposition 1.26 (b), (i) is equivalent to $W_t C \subset C$ for all $t \geq 0$.

The form τ associated to (W_t) is given by

$$\begin{aligned} D(\tau) &= \{(u, v) \in \mathcal{H} \oplus \mathcal{K} \mid \lim_{t \downarrow 0} \frac{1}{t} \langle (u, v) - W_t(u, v), (u, v) \rangle < \infty\} \\ &= \{(u, v) \in \mathcal{H} \oplus \mathcal{K} \mid \lim_{t \downarrow 0} \frac{1}{t} (\langle u - e^{-tA}u, u \rangle + \langle v - e^{-tB}v, v \rangle) < \infty\} \\ &= D(\mathfrak{a}) \oplus D(\mathfrak{b}), \\ \tau((u, v)) &= \lim_{t \downarrow 0} \frac{1}{t} (\langle u - e^{-tA}u, u \rangle + \langle v - e^{-tB}v, v \rangle) = \mathfrak{a}(u) + \mathfrak{b}(v). \end{aligned}$$

By Proposition 1.10, C is invariant under (W_t) if and only if $P_C D(\tau) \subset D(\tau)$ and $\operatorname{Re} \tau(P_C(f, g), (f, g) - P_C(f, g)) \geq 0$ for all $(f, g) \in D(\mathfrak{a}) \oplus D(\mathfrak{b})$.

(i) \implies (iii): By Proposition 1.26 (b), the projection P_C satisfies

$$P_C(f_1, g) = \frac{1}{2}(f_1 + f_2, S(f_1) + g)$$

for all $f_1, f_2 \in \mathcal{H}$, $g \in \mathcal{K}^+$ such that f_1, f_2 are g -paired and $g \leq S(f_1)$.

Now assume additionally that $f_1 \in D(\mathfrak{a})$, $g \in D(\mathfrak{b})$. If C is invariant under (W_t) , then

$$P_C(f_1, g) = \frac{1}{2}(f_1 + f_2, S(f_1) + g) \in D(\mathfrak{a}) \oplus D(\mathfrak{b}),$$

hence $f_2 \in D(\mathfrak{a})$ and $S(f_1) \in D(\mathfrak{b})$. Thus, $D(\mathfrak{a})$ is a generalized ideal of $D(\mathfrak{b})$.

Let $f_1, f_2 \in D(\mathfrak{a})$ be $S(f_2)$ -paired. Then $S(f_1), S(f_2) \in D(\mathfrak{b})$ and

$$\begin{aligned} \operatorname{Re} \mathfrak{a}(f_1, f_2) &= \lim_{t \downarrow 0} \frac{1}{t} \operatorname{Re} \langle f_1 - e^{-tA}f_1, f_2 \rangle \\ &= \lim_{t \downarrow 0} \frac{1}{t} (\langle S(f_1), S(f_2) \rangle - \operatorname{Re} \langle e^{-tA}f_1, f_2 \rangle) \\ &\geq \lim_{t \downarrow 0} \frac{1}{t} (\langle S(f_1), S(f_2) \rangle - \langle e^{-tB}S(f_1), S(f_2) \rangle) \\ &= \lim_{t \downarrow 0} \frac{1}{t} \langle Sf_1 - e^{-tB}S(f_1), S(f_2) \rangle \\ &= \mathfrak{b}(S(f_1), S(f_2)). \end{aligned}$$

For the remainder of the proof we assume that \mathcal{K}^+ is a self-dual isotone projection cone.

(iii) \implies (i): First we show that $P_C D(\tau) \subset D(\tau)$. For that purpose let $(f_1, g) \in D(\tau)$. By Proposition 1.26 (d), the projection $P_C(f_1, g)$ is given by

$$P_C(f_1, g) = \frac{1}{2}(f_2, (S(f_1) \vee g + g)^+),$$

where $f_2 \in \mathcal{H}$ such that f_1, f_2 are $(S(f_1) \wedge g + S(f_1))^+$ -paired.

By isotonicity,

$$S(f_2) = (S(f_1) \wedge g + S(f_1))^+ \leq (2S(f_1))^+ = 2S(f_1).$$

Since $D(\mathfrak{a})$ is generalized ideal of $D(\mathfrak{b})$, this inequality implies $f_2 \in D(\mathfrak{a})$.

Furthermore, $S(f_1) \in D(\mathfrak{b})$ once again since $D(\mathfrak{a})$ is a generalized ideal in $D(\mathfrak{b})$ and therefore $\frac{1}{2}(S(f_1) \vee g + g)^+ \in D(\mathfrak{b})$ since $D(\mathfrak{b})$ is a sublattice of \mathcal{K} by Lemma 1.13. Hence, $P_C(f_1, g) \in D(\tau)$.

Let $g_2 = \frac{1}{2}(S(f_1) \vee g + g)^+$. By (iii) and Lemma 1.20 we have

$$\begin{aligned} \operatorname{Re} \tau(P_C(f_1, g), (1 - P_C)(f_1, g)) &= \operatorname{Re} \mathfrak{a} \left(\frac{1}{2}f_2, f_1 - \frac{1}{2}f_2 \right) + \mathfrak{b}(g_2, g - g_2) \\ &\geq \mathfrak{b} \left(\frac{1}{2}S(f_2), S(f_1 - \frac{1}{2}f_2) \right) + \mathfrak{b}(g_2, g - g_2) \\ &= \mathfrak{b} \left(\frac{1}{2}S(f_2), S(f_1) - \frac{1}{2}S(f_2) \right) + \mathfrak{b}(g_2, g - g_2) \\ &= (\mathfrak{b} \oplus \mathfrak{b}) \left(\frac{1}{2}(S(f_2), g_2), (S(f_1), g) - (S(f_2), g_2) \right). \end{aligned}$$

Now Corollary 1.28 implies $(S(f_2), g_2) = \pi(S(f_1), g)$ and

$$(\mathfrak{b} \oplus \mathfrak{b})(\pi(S(f_1), g), (1 - \pi)(S(f_1), g)) \geq 0.$$

Therefore,

$$\operatorname{Re} \tau(P_C(f_1, g), (1 - P_C)(f_1, g)) \geq 0.$$

In the light of our preparatory work this means that (e^{-tA}) is dominated by (e^{-tB}) . \square

Remark: • A first version of this theorem was given independently by Simon (see [43], Thm. 5.1) for operators on L^2 -spaces and by Hess, Schrader, Uhlenbrock [14] in the setting of symmetrizations between abstract Hilbert spaces. Both did not offer a characterization purely in terms of forms, but the following inequality

$$\operatorname{Re} \langle g \overline{\operatorname{sgn} f}, Af \rangle \geq \mathfrak{b}(|f|, g)$$

for $f \in D(A)$, $g \in D(\mathfrak{b})^+$.

- The characterization in terms of the associated forms was first given by Ouhabaz [34] for semigroups on L^2 -spaces (vector-valued for \mathcal{H}).
- Although phrased in the terminology of [14], our proof is closer related to those in [34] and Manavi, Vogt, Voigt [31] in that we take Proposition 1.10 as a main ingredient. This approach has also the advantage that (iii) can be phrased in terms of forms and does not involve the domain of the generator. This is not only essential for our application, but also seems conceptually better fitting.
- The work [31] is concerned with some further ramifications of this theorem in the case of L^2 -spaces, allowing not necessarily densely defined, sectorial forms and giving criteria on cores. We believe that those carry over to our more abstract setting, however, this was not the focus of this article and we will not need such criteria later.
- If \mathfrak{a} is dominated by \mathfrak{b} , $f_1 \in D(\mathfrak{a})$ and $g \in D(\mathfrak{b})_+$ with $g \leq S(f_1)$, there is an $f_2 \in D(\mathfrak{a})$ such that f_1, f_2 are g -paired since $D(\mathfrak{a})$ is a generalized ideal of $D(\mathfrak{b})$. The condition $g \leq S(f_1)$ cannot be dropped, as the following example shows: Let \mathcal{E} be the standard energy form on \mathbb{R}^n , that is,

$$D(\mathcal{E}) = H^1(\mathbb{R}^n), \mathcal{E}(u) = \int_{\mathbb{R}^n} |\nabla u|^2 dx.$$

Then \mathcal{E} is positive, hence dominated by itself. Let $f_1 \in C_c^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp} f_1 \subset [-1, 1]$ and $f_1|_{[0,1]} \geq 0$, $f_1|_{[-1,0]} \leq 0$. Let $g \in C_c^\infty(\mathbb{R}^n)$, $g \geq 0$, $g|_{[-2,2]} = 1$. If f_1 and f_2 are g -paired, then $f_2(x) = g(x) \operatorname{sgn} f_1(x) = \operatorname{sgn}(x)$ for all $x \in [-1, 1]$. Hence, $f_2 \notin H^1(\mathbb{R}^n)$.

- It is interesting to note that a stronger assumption on \mathcal{K}^+ is needed for the implication (iii) \implies (i) while the theorem in [14] works without further assumption on \mathcal{K}^+ . However, it is obvious that our proof strategy strongly relies on the fact that \mathcal{K}^+ is an isotone projection cone and \mathcal{K} therefore a Riesz space. The proof in [14] also does not carry over to our situation as far as we can see.

For convenience we reformulate the above theorem for the case of L^2 -spaces. There, we will need the concept of a Hermitian vector bundle (see Example 1.18).

Corollary 1.34: *Let X be a topological space, m a Borel measure on X and E a Hermitian vector bundle over X , A (resp. B) a lower semibounded, self-adjoint operator on $L^2(X, m; E)$ (resp. $L^2(X, m)$) and \mathfrak{a} (resp. \mathfrak{b}) the associated form. Then the following are equivalent:*

- (i) *The semigroup (e^{-tA}) is dominated by (e^{-tB}) .*
- (ii) *The domain $D(\mathfrak{a})$ is a generalized ideal of $D(\mathfrak{b})$ and*

$$\operatorname{Re} \mathfrak{a}(u, \tilde{u}) \geq \mathfrak{b}(|u|, |\tilde{u}|)$$

holds for all $u, \tilde{u} \in D(\mathfrak{a})$ satisfying $\langle u(x), \tilde{u}(x) \rangle_x = |u(x)|_x |\tilde{u}(x)|_x$ for almost all $x \in X$.

Proof. It suffices to show that $u, \tilde{u} \in L^2(X, m; E)$ are paired if and only if $\langle u(x), \tilde{u}(x) \rangle_x = |u(x)|_x |\tilde{u}(x)|_x$ holds for almost all $x \in X$.

So, let us assume first that $\langle u(x), \tilde{u}(x) \rangle_x = |u(x)|_x |\tilde{u}(x)|_x$ holds for almost all $x \in X$. Then,

$$\langle u, \tilde{u} \rangle_{L^2(X, m; E)} = \int_X \langle u(x), \tilde{u}(x) \rangle_x dm(x) = \int_X |u(x)|_x |\tilde{u}(x)|_x dm(x) = \langle |u|, |\tilde{u}| \rangle_{L^2(X, m)}.$$

Hence u and \tilde{u} are paired.

Conversely, assume that u and \tilde{u} are paired. Then

$$0 = \langle |u|, |\tilde{u}| \rangle_{L^2(X, m)} - \langle u, \tilde{u} \rangle_{L^2(X, m; E)} = \int_X (|u(x)|_x |\tilde{u}(x)|_x - \langle u(x), \tilde{u}(x) \rangle_x) dm(x).$$

Moreover, the integrand is positive by Cauchy-Schwarz inequality. Therefore it must be zero almost everywhere. \square

Example 1.35: Let (X, \mathcal{B}, m) be a measure space and (Q_t) a positivity preserving semigroup on $L^2(X, m)$ with associated form \mathfrak{b} . In Example 1.23 it was remarked that (Q_t) is dominated by itself, hence $D(\mathfrak{b})$ is a generalized ideal in itself and

$$\operatorname{Re} \mathfrak{b}(u, v) \geq \mathfrak{b}(|u|, |v|)$$

holds for all $u, v \in D(\mathfrak{b})$ satisfying $u\bar{v} = |u||v|$. Now let $V: X \rightarrow [0, \infty)$ be measurable and define the form \mathfrak{a} via

$$D(\mathfrak{a}) = \{u \in D(\mathfrak{b}) \mid V^{\frac{1}{2}}u \in L^2(X, m)\}, \quad \mathfrak{a}(u) = \mathfrak{b}(u) + \int_X V|u|^2 dm.$$

In the light of Example 1.23, every positive form dominates a form, namely itself. Combined with the following corollary this gives a full characterization of the forms that can occur as dominating forms (if the positive cone is a self-dual, isotone projection cone).

Corollary 1.36 (Automatic positivity of dominating forms): *Let $\mathcal{K}^+ \subset \mathcal{K}$ be a self-dual, isotone projection cone, $S: \mathcal{H} \rightarrow \mathcal{K}^+$ a symmetrization and A (resp. B) a lower semibounded, self-adjoint operator on \mathcal{H} (resp. \mathcal{K}), and \mathfrak{a} (resp. \mathfrak{b}) the associated form. If \mathfrak{a} is dominated by \mathfrak{b} , then \mathfrak{b} is a positive form.*

Proof. By the characterization of domination (Theorem 1.33), (e^{-tA}) is dominated by (e^{-tB}) . Now, the result follows from Corollary 1.25. \square

2. A CRITERION FOR FORM UNIQUENESS

In this section we present the main theorem of this article, which allows one to transfer form uniqueness of a dominating form to that of the dominated form. Having set up the stage in the last chapter, we only need one further technical lemma to provide its proof. This lemma might also be of interest in other situations.

Lemma 2.1: *Let \mathcal{K} be a real Hilbert space, $\mathcal{K}^+ \subset \mathcal{K}$ a self-dual isotone projection cone, $(\mathfrak{b}, D(\mathfrak{b}))$ a closed, positive form on \mathcal{K} and $D_{\mathfrak{b}} \subset D(\mathfrak{b})$ a dense ideal. If $v \in D(\mathfrak{b})^+$, then there is a sequence (v_n) in $D_{\mathfrak{b}}$ such that $0 \leq v_n \leq v$ and $\|v_n - v\|_{\mathfrak{b}} \rightarrow 0$.*

Proof. Let $v \in D(\mathfrak{b})^+$. Since $D_{\mathfrak{b}} \subset D(\mathfrak{b})$ is dense, there is a sequence \tilde{v}_n in $D_{\mathfrak{b}}$ such that $\|\tilde{v}_n - v\|_{\mathfrak{b}} \rightarrow 0$. By Lemma 1.13, $D(\mathfrak{b})$ is a sublattice of \mathcal{K} , hence $\tilde{v}_n^+ \wedge v \in D(\mathfrak{b})$. From the inequality

$$0 \leq \tilde{v}_n^+ \wedge v \leq v_n^+ \leq |\tilde{v}_n|$$

it follows that $\tilde{v}_n^+ \wedge v \in D_{\mathfrak{b}}$ since $D_{\mathfrak{b}} \subset D(\mathfrak{b})$ is an ideal.

Let $-\lambda < 0$ be a lower bound for \mathfrak{b} . Then Lemma 1.13 implies that

$$\|\tilde{v}_n^+ \wedge v\|_{\mathfrak{b}}^2 = \mathfrak{b}_{1+\lambda}(\tilde{v}_n^+ \wedge v) \leq \mathfrak{b}_{1+\lambda}(\tilde{v}_n^+) + \mathfrak{b}_{1+\lambda}(v) \leq \mathfrak{b}_{1+\lambda}(\tilde{v}_n) + \mathfrak{b}_{1+\lambda}(v) = \|\tilde{v}_n\|_{\mathfrak{b}}^2 + \|v\|_{\mathfrak{b}}^2.$$

Since (\tilde{v}_n) is convergent in $(D(\mathfrak{b}), \langle \cdot, \cdot \rangle_{\mathfrak{b}})$, it is in particular bounded, and the above inequality shows that $(\tilde{v}_n^+ \wedge v)$ is bounded as well. By the Banach-Saks Theorem (cf. [47], Satz V.3.8) there is a subsequence (\tilde{v}_{n_k}) and an element $\tilde{v} \in D(\mathfrak{b})$ such that

$$v_N := \frac{1}{N} \sum_{k=1}^N \tilde{v}_{n_k}^+ \wedge v \xrightarrow{\|\cdot\|_{\mathfrak{b}}} \tilde{v}, \quad N \rightarrow \infty.$$

Obviously, $v_N \in D_{\mathfrak{b}}$ and $0 \leq v_N \leq v$ for all $N \in \mathbb{N}$. Moreover, we have

$$|v - \tilde{v}_n| = (v \vee \tilde{v}_n) - (v \wedge \tilde{v}_n) \geq v - v \wedge \tilde{v}_n$$

and

$$v \wedge \tilde{v}_n^+ = v \wedge (\tilde{v}_n \vee 0) = (v \wedge \tilde{v}_n) \vee (v \wedge 0) \geq v \wedge \tilde{v}_n,$$

hence

$$|v - \tilde{v}_n| \geq v - v \wedge \tilde{v}_n^+ \geq 0.$$

By Lemma 1.8, this inequality implies

$$\|v - v \wedge \tilde{v}_n^+\|_{\mathcal{K}} \leq \|v - \tilde{v}_n\|_{\mathcal{K}} \rightarrow 0$$

and therefore also

$$\|v - v_N\|_{\mathcal{K}} \rightarrow 0.$$

Thus, we finally infer $\tilde{v} = v$. □

Theorem 2.2: *Let \mathcal{H} be a Hilbert space, \mathcal{K} a real Hilbert space, $\mathcal{K}^+ \subset \mathcal{K}$ a self-dual isotone projection cone, and $S: \mathcal{H} \rightarrow \mathcal{K}^+$ a symmetrization.*

Let $(\mathfrak{a}, D(\mathfrak{a}))$ be a closed form in \mathcal{H} , $(\mathfrak{b}, D(\mathfrak{b}))$ a closed form in \mathcal{K} , and $D_{\mathfrak{a}} \subset D(\mathfrak{a}), D_{\mathfrak{b}} \subset D(\mathfrak{b})$ ideals such that the following conditions hold:

- \mathfrak{a} is dominated by \mathfrak{b}
- $D_{\mathfrak{b}}^+ \cap S(D(\mathfrak{a})) \subset S(D_{\mathfrak{a}})$

If $D_{\mathfrak{b}}$ is a form core for \mathfrak{b} , then $D_{\mathfrak{a}}$ is a form core for \mathfrak{a} .

Proof. Let $-\lambda < 0$ be a common lower bound for \mathfrak{a} and \mathfrak{b} . As \mathfrak{a} is closed, $D(\mathfrak{a})$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{a}} = (1 + \lambda)\langle \cdot, \cdot \rangle_{\mathcal{H}} + \mathfrak{a}(\cdot, \cdot)$ and analogously for \mathfrak{b} .

We show that $D_{\mathfrak{a}} \subset D(\mathfrak{a})$ is dense with respect to $\|\cdot\|_{\mathfrak{a}}$ by proving that $D_{\mathfrak{a}}^{\perp} = \{0\}$ in $(D(\mathfrak{a}), \langle \cdot, \cdot \rangle_{\mathfrak{a}})$. For this purpose, let $h \in D(\mathfrak{a})$ such that

$$0 = \langle h, u \rangle_{\mathfrak{a}} = (1 + \lambda)\langle h, u \rangle + \mathfrak{a}(h, u)$$

for all $u \in D_{\mathfrak{a}}$.

Next take $v \in D_{\mathfrak{b}}^+$ such that $v \leq S(h)$. Since \mathfrak{a} is dominated by \mathfrak{b} , $D(\mathfrak{a})$ is a generalized ideal of $D(\mathfrak{b})$. Hence there is an $\tilde{h} \in D(\mathfrak{a})$ such that h, \tilde{h} are v -paired. In particular, $S(\tilde{h}) = v \in D_{\mathfrak{b}}^+ \cap S(D(\mathfrak{a})) \subset S(D_{\mathfrak{a}})$, and since $D_{\mathfrak{a}}$ is an ideal in $D(\mathfrak{a})$, $\tilde{h} \in D_{\mathfrak{a}}$. Moreover, since \mathfrak{a} is dominated by \mathfrak{b} , $S(h) \in D(\mathfrak{b})$ and

$$(*) \quad 0 = (1 + \lambda)\langle h, \tilde{h} \rangle + \operatorname{Re} \mathfrak{a}(h, \tilde{h}) \geq (1 + \lambda)\langle S(h), v \rangle + b(S(h), v).$$

As \mathfrak{b} dominates \mathfrak{a} , the form \mathfrak{b} is positive by Corollary 1.36. Hence, the assumption of Lemma 2.1 is satisfied and there exists a sequence (v_n) in $D_{\mathfrak{b}}$ such that $0 \leq v_n \leq S(h)$ and $\|v_n - S(h)\|_{\mathfrak{b}} \rightarrow 0$. Applying inequality $(*)$ to $v = v_N$ we obtain

$$0 \geq (1 + \lambda)\langle S(h), v_N \rangle + \mathfrak{b}(S(h), v_N) = \langle S(h), v_N \rangle_{\mathfrak{b}} \rightarrow \|S(h)\|_{\mathfrak{b}}^2.$$

Hence $S(h) = 0$ and therefore also $h = 0$. Thus, $D_{\mathfrak{a}}^{\perp} = \{0\}$. \square

Remark: • In applications, the situation will often be as follows: We are given forms \mathfrak{a}_0 on $D_{\mathfrak{a}}$, \mathfrak{b}_0 on $D_{\mathfrak{b}}$ (usually not closed) and minimal extensions \mathfrak{a}_{\min} , \mathfrak{b}_{\min} (the closures of \mathfrak{a}_0 , \mathfrak{b}_0) and maximal extensions \mathfrak{a}_{\max} , \mathfrak{b}_{\max} .

If $\mathfrak{b}_{\min} = \mathfrak{b}_{\max}$, then the theorem yields $\mathfrak{a}_{\min} = \mathfrak{a}_{\max}$.

This is discussed in detail in Chapter 3.

- If $\mathcal{H} = \mathcal{K}$ and $S = |\cdot|$ and $D_{\mathfrak{a}} = D_{\mathfrak{b}}$, then the condition $D_{\mathfrak{b}}^+ \subset S(D_{\mathfrak{a}})$ is automatically satisfied.
- In the light of Theorem 1.33, the condition that \mathfrak{a} is dominated by \mathfrak{b} can also be phrased in terms of the associated semigroups or resolvents.
- In the case of L^2 -spaces, the application of the Banach-Saks Theorem in the proof can be replaced by the fact that the L^2 -convergent sequence (v_n) has an a.e. convergent subsequence.
- Note that we used the main result of the previous section, Theorem 1.26, in form of Corollary 1.36 in the proof.

We now turn to a corollary that contains the concrete situation of our applications in the next chapter. There, we consider a topological space X and μ a Borel measure on X . In this situation, we denote by $L_c^\infty(X, \mu)$ the space of essentially bounded functions that vanish outside a compact set. Whenever $E \rightarrow X$ is a Hermitian vector bundle over X , we denote by $L_c^\infty(X, \mu; E)$ the space of essentially bounded sections in E that vanish outside a compact set.

Corollary 2.3: *Let X be a topological space, μ a Borel measure on X and $E \rightarrow X$ a Hermitian vector bundle over X . Assume that $(\mathfrak{b}, D(\mathfrak{b}))$ is a closed form in $L^2(X, \mu)$ and $(\mathfrak{a}, D(\mathfrak{a}))$ a closed form in $L^2(X, \mu; E)$ that is dominated by \mathfrak{b} . If $D(\mathfrak{b}) \cap L_c^\infty(X, \mu)$ is a form core for \mathfrak{b} , then $D(\mathfrak{a}) \cap L_c^\infty(X, \mu; E)$ is a form core for \mathfrak{a} .*

Proof. We will apply Theorem 2.2 to $D_{\mathfrak{a}} = L_c^\infty(X, \mu; E) \cap D(\mathfrak{a})$ and $D_{\mathfrak{b}} = L_c^\infty(X, \mu) \cap D(\mathfrak{b})$. It is obvious that these are ideals in $D(\mathfrak{a})$ and $D(\mathfrak{b})$ respectively.

Now let $g \in D_{\mathfrak{b}}^+ \cap |D(\mathfrak{a})|$. Then there is an $f \in D(\mathfrak{a})$ such that $|f| = g \in L_c^\infty(X, \mu)$. Thus, $f \in L_c^\infty(X, \mu; E) \cap D(\mathfrak{a})$ and $g = |f| \in |D_{\mathfrak{a}}|$. \square

Remark: • We tacitly assumed that $L^2(X, \mu)$ is the space of real-valued L^2 -functions in order to apply Theorem 2.2, whereas $L^2(X, \mu; E)$ may be viewed either as real or as complex. We will adopt this convention also for the applications of this corollary in the next chapter.

- If $(\mathfrak{b}, D(\mathfrak{b}))$ is a regular Dirichlet form, $L_c^\infty(X, \mu) \cap D(\mathfrak{b})$ is a form core for \mathfrak{b} . Indeed, $C_c(X) \cap D(\mathfrak{b}) \subset L_c^\infty(X, \mu) \cap D(\mathfrak{b})$ is dense in $D(\mathfrak{b})$ by definition. Note, however, that we do not use the second Beurling-Deny criterion in our reasoning.
- In the applications in this article we will only encounter trivial vector bundles and $L^2(X, \mu; E)$ can be identified with $L^2(X, \mu; \mathbb{C}^n)$ via the trivialization. However, this is not necessarily the case in other possible applications, for example if \mathfrak{a} is the form associated with the Hodge-de Rham Laplacian on differential forms on a Riemannian manifold.

In applications to manifolds one is in an even more regular situation. More specifically, in the smooth case, one is usually interested in the closure of the form defined on smooth functions (sections) as minimal form. We make the following definition adapted to this situation.

Definition 2.4: Let M be a Riemannian manifold and $E \rightarrow M$ a smooth Hermitian vector bundle. A form \mathbf{a} on $L^2(M; E)$ is called smoothly inner regular if $\Gamma_c^\infty(M; E) \cap D(\mathbf{a})$ is dense in $D(\mathbf{a}) \cap L_c^\infty(M; E)$ with respect to $\|\cdot\|_{\mathbf{a}}$.

From the definition of smooth inner regularity and the above corollary, the following corollary can easily be deduced.

Corollary 2.5: Let M be a Riemannian manifold and $E \rightarrow M$ a smooth Hermitian vector bundle. Let \mathbf{b} be a closed form on $L^2(M)$ and \mathbf{a} a closed, smoothly inner regular form on $L^2(M; E)$ that is dominated by \mathbf{b} . If $C_c^\infty(M) \cap D(\mathbf{b})$ is a form core for \mathbf{b} , then $\Gamma_c^\infty(M; E) \cap D(\mathbf{a})$ is a form core for \mathbf{a} .

Remark: The preceding two corollaries concern bundles E over an underlying topological space X . The involved Hilbert spaces are the L^2 -space of the underlying space and the L^2 -space of the bundle. This L^2 -space of the bundle can also be considered as a direct integral of Hilbert spaces over X . In fact, it is easily possible to generalize the Corollaries to direct integrals over a topological space X (provided that there exists a suitable symmetrization). We refrain from giving a detailed statement as we do not need this for the purposes of this article.

3. APPLICATIONS

3.1. Magnetic Schrödinger forms on graphs. In this section we will study discrete analogs of the Laplacian respectively magnetic Schrödinger operators in Euclidean space. Analysis on graphs has been an active field of research in recent years and uniqueness of extensions of operators respectively forms on graphs have been intensively studied. We just point to [10, 11] for non-magnetic forms and the recent series of works by Milatovic and Truc [29, 30] for magnetic forms as a few examples.

Compared to the Euclidean case, the discrete setting allows more clarity in the presentation as some mere technical complications do not appear. In particular, Corollary 2.3 can be applied directly since $L_c^\infty(X)$ and $C_c(X)$ coincide for discrete spaces.

We will start with some basic definitions, including those of magnetic Schrödinger forms on graphs (Definitions 3.3 and 3.4), essentially following [18], [19] regarding graphs and Dirichlet forms over discrete spaces and [30] regarding vector bundles over graphs and magnetic Schrödinger operators. Then we show that the form with magnetic field is dominated by the form without magnetic field (Proposition 3.6) before we finally give the uniqueness result (Theorem 3.8) and discuss some examples.

Definition 3.1 (Weighted graph): A weighted graph (X, b, c, m) consists of an (at most) countable set X , an edge weight $b: X \times X \rightarrow [0, \infty)$, a killing term $c: X \rightarrow [0, \infty)$ and a measure $m: X \rightarrow (0, \infty)$ subject to the following conditions for all $x, y \in X$:

- (b1) $b(x, x) = 0$,
- (b2) $b(x, y) = b(y, x)$,
- (b3) $\sum_{z \in X} b(x, z) < \infty$.

Observe that we do not assume our graphs to be locally finite, that is, $\{y \in X \mid b(x, y) > 0\}$ may be infinite as long as the edge weights are still summable (at each vertex). We shall regard X as a discrete topological space. Consequently, $C_c(X)$ is the space of functions on X with finite support. We regard m as a measure on the power set $\mathcal{P}(X)$ of X via

$$m(A) := \sum_{x \in A} m(x), \quad A \subset X,$$

and denote the corresponding L^2 -space by $\ell^2(X, m)$.

Any graph comes with a formal Laplacian \tilde{L} acting on

$$\{f: X \rightarrow \mathbb{R} : \sum_y b(x, y)|f(y)| < \infty \text{ for all } x \in X\}$$

by

$$\tilde{L}f(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)(f(x) - f(y) + \frac{c(x)}{m(x)}f(x))$$

for $x \in X$.

Definition 3.2 (Hermitian vector bundle): *A Hermitian vector bundle over a discrete set X is a family $F = (F_x, \langle \cdot, \cdot \rangle_x)_{x \in X}$ of finite-dimensional Hilbert spaces together with a unitary maps (called connection maps) $\Phi_{x,y}: F_y \rightarrow F_x$ for all $x, y \in X$ such that $\Phi_{x,y} = \Phi_{y,x}^{-1}$. For a Hermitian vector bundle F over X we will denote by*

$$\begin{aligned} \Gamma(X; F) &= \{u: X \rightarrow \prod_{x \in X} F_x \mid u(x) \in F_x\}, \\ \Gamma_c(X; F) &= \{u \in \Gamma(X; F) \mid \text{supp } u \text{ finite}\}, \\ \ell^2(X, m; F) &= \{u \in \Gamma(X; F) \mid \sum_{x \in X} \langle u(x), u(x) \rangle_x m(x) < \infty\} \end{aligned}$$

the space of all sections, the space of all sections with compact support and the space of all L^2 -sections. The latter becomes a Hilbert space equipped with the inner product

$$\langle \cdot, \cdot \rangle_{\ell^2(X, m; F)}: \ell^2(X, m; F) \times \ell^2(X, m; F) \rightarrow \mathbb{C}, (u, v) \mapsto \sum_{x \in X} \langle u(x), v(x) \rangle_x m(x).$$

A bundle endomorphism W of a Hermitian vector bundle F is a family of linear maps $(W(x): F_x \rightarrow F_x)_{x \in X}$.

For the remainder of the section, (X, b, c, m) is a weighted graph, F a Hermitian vector bundle over X with unitary connection Φ and W a bundle endomorphism of F that is pointwise positive, that is, $\langle W(x)v, v \rangle_x \geq 0$ for all $x \in X, v \in F_x$.

Now we can define the basic object of our interest, the magnetic Schrödinger form (with Dirichlet and Neumann boundary conditions).

Definition 3.3 (Magnetic form with Neumann boundary conditions): *For $u \in \Gamma(X; F)$ let*

$$\tilde{Q}_{\Phi, b, W}(u) = \frac{1}{2} \sum_{x, y} b(x, y) |u(x) - \Phi_{x,y} u(y)|_x^2 + \sum_x \langle W(x)u(x), u(x) \rangle_x \in [0, \infty].$$

The magnetic Schrödinger form with Neumann boundary conditions is defined via

$$\begin{aligned} D(Q_{\Phi, b, W}^{(N)}) &= \{u \in \ell^2(X, m) \mid \tilde{Q}_{\Phi, b, W}(u) < \infty\}, \\ Q_{\Phi, b, W}^{(N)}(u) &= \tilde{Q}_{\Phi, b, W}(u). \end{aligned}$$

To abridge notation, we will write $\|\cdot\|_{\Phi, b, W}$ for the form norm of $Q_{\Phi, b, W}^{(N)}$. By the same arguments as in the Dirichlet form case, the form $Q_{\Phi, b, W}^{(N)}$ is closed (see [19], Lemma 2.3).

Definition 3.4 (Magnetic form with Dirichlet boundary conditions): *The magnetic Schrödinger form with Dirichlet boundary conditions $Q_{\Phi, b, W}^{(D)}$ is the closure of the restriction of $Q_{\Phi, b, W}^{(N)}$ to $C_c(X)$.*

If $F_x = \mathbb{C}$ endowed with the standard inner product and $\Phi_{x,y} = 1$ for all $x, y \in X$, we will suppress Φ in the index and simply write $Q_{b, W}^{(D)}$ (resp. $Q_{b, W}^{(N)}$). We may also drop other indices if they are clear from the context. The interest in these forms is particularly a result of the fact that $Q_{b, c}^{(D)}$ and $Q_{b, c}^{(N)}$ are Dirichlet forms. Indeed, all regular Dirichlet forms over a discrete measure space are of the form $Q_{b, c}^{(D)}$ for some graph (X, b, c) (cf. [19], Lemma 2.2). This is one motivation to study also graphs that are not locally finite.

For our subsequent considerations we also note that the generators of both $Q_{b, c}^{(D)}$ and $Q_{b, c}^{(N)}$ are restrictions of \tilde{L} (see [10]). So, if the restriction

$$L_0 := \tilde{L}|_{C_c(X)}$$

of \tilde{L} to $C_c(X)$ maps into $\ell^2(X, m)$ and is essentially self-adjoint then $Q_{b,c}^{(D)} = Q_{b,c}^{(N)}$ follows.

As a next step to establish criteria for $Q_\Phi^{(N)} = Q_\Phi^{(D)}$ we will show that the form with magnetic field is dominated by the non-magnetic form. First we prove an easy technical lemma.

Lemma 3.5: *Let V be a Hilbert space, $a, b \in V$, and $\alpha, \beta \geq 0$ with $\alpha \leq \|a\|$, $\beta \leq \|b\|$. Define*

$$\tilde{a} = \begin{cases} \frac{\alpha}{\|a\|}a & : a \neq 0 \\ 0 & : a = 0 \end{cases}$$

and likewise \tilde{b} . Then

$$\|\tilde{a} - \tilde{b}\|^2 \leq |\alpha - \beta|^2 + \|a - b\|^2.$$

Proof. If $a = 0$ or $b = 0$, the inequality is obvious. Hence assume that $a, b \neq 0$. In the following computation we use the inequality $2\lambda\mu \leq \lambda^2 + \mu^2$ for $\lambda, \mu \in \mathbb{R}$.

$$\begin{aligned} \|\tilde{a} - \tilde{b}\|^2 &= \|\tilde{a}\|^2 + \|\tilde{b}\|^2 - 2\operatorname{Re}\langle \tilde{a}, \tilde{b} \rangle \\ &= \alpha^2 + \beta^2 - 2\operatorname{Re}\langle \tilde{a}, \tilde{b} \rangle \\ &= |\alpha - \beta|^2 + 2\alpha\beta - 2\operatorname{Re}\langle \tilde{a}, \tilde{b} \rangle \\ &= |\alpha - \beta|^2 + 2\frac{\alpha\beta}{\|a\|\|b\|}(\|a\|\|b\| - \operatorname{Re}\langle a, b \rangle) \\ &\leq |\alpha - \beta|^2 + 2\|a\|\|b\| - 2\operatorname{Re}\langle a, b \rangle \\ &\leq |\alpha - \beta|^2 + \|a\|^2 + \|b\|^2 - 2\operatorname{Re}\langle a, b \rangle \\ &= |\alpha - \beta|^2 + \|a - b\|^2 \end{aligned}$$

□

We will now prove that the magnetic form is dominated by the form without magnetic field. In the form of a pointwise Kato's inequality this result was given in [30], Lemma 3.3.

Proposition 3.6: *Assume that $\langle W(x)u(x), u(x) \rangle_x \geq c(x)|u(x)|^2$ for all $x \in X$, $u(x) \in F_x$. Then $Q_{\Phi,b,W}^{(N)}$ is dominated by $Q_{b,c}^{(N)}$.*

Proof. By the characterization of domination, Corollary 1.34, it suffices to show that $D(Q_{\Phi,b,W}^{(N)})$ is a generalized ideal in $D(Q_{b,c}^{(N)})$ and that

$$\operatorname{Re} Q_{\Phi,b,W}^{(N)}(u, \tilde{u}) \geq Q_{b,c}^{(N)}(|u|, |\tilde{u}|)$$

holds for all $u, \tilde{u} \in D(Q_{\Phi,b,W}^{(N)})$ such that $\langle u(x), \tilde{u}(x) \rangle_x = |u(x)||\tilde{u}(x)|$ for all $x \in X$.

First, let $u \in D(Q_{\Phi,b,W}^{(N)})$. Then $|u| \in \ell^2(X, m)$ and

$$\begin{aligned} \tilde{Q}_{\Phi,b,W}(u) &= \frac{1}{2} \sum_{x,y} b(x,y) |u(x) - \Phi_{x,y}u(y)|^2 + \sum_x \langle W(x)u(x), u(x) \rangle \\ &\geq \frac{1}{2} \sum_{x,y} b(x,y) ||u(x)| - |u(y)||^2 + \sum_x c(x)|u(x)|^2 \\ &= \tilde{Q}_{b,c}(|u|), \end{aligned}$$

hence $|u| \in D(Q_{b,c}^{(N)})$.

Next let $v \in D(Q_{b,c}^{(N)})$ with $0 \leq v \leq |u|$. Obviously, $\|v \operatorname{sgn} u\|_{\ell^2} \leq \|v\|_{\ell^2}$, thus $v \operatorname{sgn} u \in \ell^2(X, m; F)$. Applying Lemma 3.5 to $V = F_x$, $a = u(x)$, $b = \Phi_{x,y}u(y)$, $\alpha = v(x)$, $\beta = v(y)$, we obtain

$$|v(x) \operatorname{sgn} u(x) - \Phi_{x,y}v(y) \operatorname{sgn} u(y)|^2 \leq |v(x) - v(y)|^2 + |u(x) - \Phi_{x,y}u(y)|^2.$$

Summation over x, y implies

$$\tilde{Q}_{\Phi,b,0}(v \operatorname{sgn} u) \leq Q_{b,0}^{(N)}(v) + Q_{\Phi,b,0}^{(N)}(u).$$

Furthermore,

$$\begin{aligned} \sum_x \langle W(x)v(x) \operatorname{sgn} u(x), v(x) \operatorname{sgn} u(x) \rangle &\leq \sum_x |u(x)|^2 \langle W(x) \operatorname{sgn} u(x), \operatorname{sgn} u(x) \rangle \\ &= \sum_x \langle W(x)u(x), u(x) \rangle, \end{aligned}$$

hence

$$\tilde{Q}_{\Phi,b,W}(v \operatorname{sgn} u) \leq Q_{b,0}^{(N)}(v) + Q_{\Phi,b,0}^{(N)}(u) + \sum_x c(x)|u(x)|^2 \leq Q_{b,c}^{(N)}(v) + Q_{\Phi,b,W}^{(N)}(u),$$

that is, $v \operatorname{sgn} u \in D(Q_{\Phi,b,W}^{(N)})$.

Let $u, \tilde{u} \in D(Q_{\Phi,b,W}^{(N)})$ such that $\langle u(x), \tilde{u}(x) \rangle_x = |u(x)||\tilde{u}(x)|$ for all $x \in X$. Then we have

$$\begin{aligned} &\operatorname{Re} \langle u(x) - \Phi_{x,y}u(y), \tilde{u}(x) - \Phi_{x,y}\tilde{u}(y) \rangle \\ &= \operatorname{Re}(\langle u(x), \tilde{u}(x) \rangle - \langle u(x), \Phi_{x,y}\tilde{u}(y) \rangle - \langle \Phi_{x,y}u(y), \tilde{u}(x) \rangle + \langle u(y), \tilde{u}(y) \rangle) \\ &= |u(x)||\tilde{u}(x)| + |u(y)||\tilde{u}(y)| - \operatorname{Re} \langle u(x), \Phi_{x,y}\tilde{u}(y) \rangle - \operatorname{Re} \langle \Phi_{x,y}u(y), \tilde{u}(x) \rangle \\ &\geq |u(x)||\tilde{u}(x)| + |u(y)||\tilde{u}(y)| - |u(x)||\tilde{u}(y)| - |u(y)||\tilde{u}(x)| \\ &= (|u(x)| - |u(y)|)(|\tilde{u}(x)| - |\tilde{u}(y)|) : \end{aligned}$$

After multiplication with $b(x, y)$ and summation over $x, y \in X$ we get

$$\operatorname{Re} Q_{\Phi,b,W}^{(N)}(u, \tilde{u}) \geq Q_{b,c}^{(N)}(|u|, |\tilde{u}|). \quad \square$$

Corollary 3.7: *The form $Q_{b,c}^{(N)}$ is dominated by $Q_{\Phi,b,W}^{(N)}$.*

Proof. This follows from Proposition 3.6 by taking $F_x = \mathbb{C}$, $W(x) = c(x)$ and $\Phi_{x,y} = 1$ for all $x, y \in X$. \square

Having proven the domination property, the announced main result of this section is now an easy consequence of Corollary 2.3. In a very informal way it says that adding a magnetic and electric field does not disturb the form uniqueness.

Theorem 3.8: *Assume that $\langle W(x)u(x), u(x) \rangle \geq c(x)|u(x)|^2$ for all $x \in X$, $u(x) \in F_x$. If $Q_{b,c}^{(D)} = Q_{b,c}^{(N)}$, then $Q_{\Phi,b,W}^{(D)} = Q_{\Phi,b,W}^{(N)}$.*

Proof. We have proven in Proposition 3.6 that $Q_{\Phi,b,W}^{(N)}$ is dominated by $Q_{b,c}^{(N)}$. An application of Corollary 2.3 for $\mathfrak{a} = Q_{\Phi,b,W}^{(N)}$ and $\mathfrak{b} = Q_{b,c}^{(N)}$ yields the claim. \square

To apply the theorem, we need $Q_{b,c}^{(D)} = Q_{b,c}^{(N)}$. There are quite a few conditions under which this equality holds. The first were phrased in terms of the measure m and the combinatorial graph structure:

Example 3.9 (See [19] for details and proofs): If $\tilde{L}C_c(X) \subset \ell^2(X, m)$ and $\sum_{n=1}^{\infty} m(x_n) = \infty$ for any sequence (x_n) in X such that $b(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$, then L_0 is essentially self-adjoint and all form extensions of $Q^{(D)}$ coincide with $Q^{(D)}$. The given conditions are in particular satisfied if $\inf_{x \in X} m(x) > 0$.

It turns out that the concept of intrinsic pseudo metrics (introduced for general not necessarily local Dirichlet forms in [7]) provides a suitable framework for many conditions for uniqueness of form extensions. Here, a pseudo metric $d: X \times X \rightarrow [0, \infty)$ is called intrinsic if

$$\frac{1}{m(x)} \sum_y b(x, y) d(x, y)^2 \leq 1$$

for all $x \in X$. A pseudo metric d is said to have finite jump size if there is an $s \in \mathbb{R}$ such that $b(x, y) = 0$ for all $x, y \in X$ with $d(x, y) > s$. A pseudo metric d is called a path pseudo metric if there

is a function $\sigma: X \times X \rightarrow [0, \infty)$, satisfying $\sigma(x, y) = \sigma(y, x)$ and $\sigma(x, y) > 0$ iff $b(x, y) > 0$ for all $x, y \in X$, such that

$$d(x, y) = d_\sigma(x, y) := \inf_\gamma \sum_{k=1}^n \sigma(x_{k-1}, x_k)$$

where the infimum is taken over all paths (x_0, \dots, x_n) connecting x and y . An intrinsic path pseudo metric d_σ is called strongly intrinsic if

$$\frac{1}{m(x)} \sum_y b(x, y) \sigma(x, y)^2 \leq 1$$

for all $x \in X$.

The following conditions are taken from [11], Theorem 1 and 2. Further examples can be found there.

Example 3.10: Let d be an intrinsic pseudo metric on (X, b, c, m) . If the weighted degree function

$$\text{Deg}: X \rightarrow [0, \infty), \text{Deg}(x) = \frac{1}{m(x)} \left(\sum_{y \in X} b(x, y) + c(x) \right)$$

is bounded on the combinatorial neighborhood of each distance ball, then $Q^{(D)} = Q^{(N)}$.

Example 3.11: If $(X, b, 0, m)$ is locally finite and there is an intrinsic path metric d such that (X, d) is metrically complete, then L_0 is essentially self-adjoint and consequently $Q^{(D)} = Q^{(N)}$.

The following condition is given in [12], Lemma 2.5, and [1], Theorem 1. Its connection to intrinsic metrics is discussed in [12], Theorem 2.7.

Example 3.12: If the graph is *complete*, then $Q^{(D)} = Q^{(N)}$. Here, completeness means that there is a non-decreasing sequence (η_k) in $C_c(X)$ such that $\eta_k \rightarrow 1$ pointwise and

$$\frac{1}{m(x)} \sum_y b(x, y) |\eta_k(x) - \eta_k(y)|^2 \leq \frac{1}{k}$$

for all $x \in X$, $k \in \mathbb{N}$.

Remark: The previous example shows the strength of our method as it has not been treated in earlier works. Notice that we have only $Q^{(D)} = Q^{(N)}$ and not the stronger assertion that L_0 is essentially self-adjoint as in the examples before.

3.2. Magnetic Schrödinger forms on domains in Euclidean space. In this section we will show that coincidence of the minimal and maximal form for the Laplacian defined below implies the coincidence of minimal and maximal form of a magnetic Schrödinger operator on domains in Euclidean space. These operators and questions of uniqueness of extension have been extensively studied; for background information and early treatments we refer the reader to [16, 45].

Contrary to the discrete case treated in the last section, some more work has to be done in order to derive the uniqueness result, Theorem 3.20, from Corollary 2.5. This is due to the fact that for a form on a domain $\Omega \subset \mathbb{R}^n$, smooth inner regularity is a non-void condition.

In the first part of this section we will define the basic objects of our interest, magnetic Schrödinger forms (Definitions 3.13 and 3.15), and establish that they are closed. In the second part we show that the Schrödinger form with magnetic field is dominated by the Schrödinger form without magnetic field (Proposition 3.18). In the last part we present the announced uniqueness result (Theorem 3.20) for magnetic forms on Euclidean domains and give some examples.

In the following, $\Omega \subset \mathbb{R}^n$ shall denote a domain. For a subset $K \subset \Omega$ we write $K \subset\subset \Omega$ if \overline{K} is compact and contained in Ω .

If f is weakly differentiable, then $\nabla f = (\partial_1 f, \dots, \partial_n f)$ denotes the vector valued function with the weak derivatives of f as components.

We will use some standard function spaces as $C_c^\infty(\Omega)$, $W^{1,p}(\Omega)$ etc. By $L_c^p(\Omega)$ we denote the space of all L^p -functions that vanish outside a compact set. In addition, the corresponding vector- and matrix-valued spaces will be denoted by $L_{\text{loc}}^p(\Omega; \mathbb{C}^n)$, $L_{\text{loc}}^p(\Omega; M_n(\mathbb{R}))$ etc. By $M_n(\mathbb{R})^+$ we denote the set of real-valued, symmetric, positive matrices.

Definition 3.13 (Magnetic form with Neumann boundary conditions): *Let $a \in L_{\text{loc}}^\infty(\Omega; M_n(\mathbb{R})^+)$, $b \in L_{\text{loc}}^2(\Omega; \mathbb{R}^n)$, $V \in L_{\text{loc}}^1(\Omega)^+$. Let $D_k: L^2(\Omega) \rightarrow \mathcal{D}'(\Omega)$, $D_k u = \partial_k u - ib_k u$, $Du = (D_1 u, \dots, D_n u)$. Assume that for every $K \subset\subset \Omega$ there is a constant $\mu_K > 0$ such that $\mu_K 1_n \leq a(x)$ for almost all $x \in K$.*

Define the magnetic form with Neumann boundary conditions via

$$D(\mathcal{E}_{a,b,V}^{(N)}) = \{u \in L^2(\Omega) \mid a^{\frac{1}{2}} Du \in L^2(\Omega; \mathbb{C}^n), V^{\frac{1}{2}} u \in L^2(\Omega)\},$$

$$\mathcal{E}_{a,b,V}^{(N)}(u, v) = \sum_{j,k=1}^n \int_{\Omega} a_{jk}(D_j u) \overline{(D_k v)} dx + \int_{\Omega} V u \bar{v} dx.$$

The form norm of $\mathcal{E}_{a,b,V}^{(N)}$ will be denoted by $\|\cdot\|_{a,b,V}$.

For the remainder of the section we shall always assume the regularity assumptions on a and b without further mentioning it. The regularity assumption on V will be weakened later in order to allow negative potentials V . We will denote by 1 the constant function $\Omega \rightarrow M_n(\mathbb{R}), x \mapsto 1_n$ (the identity matrix).

It is well-known that the form $\mathcal{E}_{a,b,V}^{(N)}$ is closed under the above regularity assumptions on a, b, V , the proof is essentially the same as that of the completeness of the standard Sobolev space $W^{1,2}(\Omega)$.

Remark: • If the coefficients are sufficiently regular, the form $\mathcal{E}_{a,b,V}$ can be viewed as form associated with the differential expression

$$\tau = - \sum_{j,k} (\partial_k - ib_k) a_{jk} (\partial_j - ib_j) + V.$$

However, the approach via forms allows us to handle the case of coefficients that are not (weakly) differentiable and for which the expression τ makes no immediate sense.

- The expression τ is elliptic with principal part

$$- \sum_{j,k} \partial_j a_{jk} \partial_k.$$

The condition $a \geq \mu_K$ on compact subsets $K \subset \Omega$ then translates to the condition of τ being locally strongly elliptic.

- In quantum mechanics, the expression $\mathcal{E}_{1,b,V}(\psi)$ is the energy of a particle with wave function ψ in an electric field with potential V and a magnetic field with magnetic potential b .

Definition 3.14 (Form small potential): *Let q be a closed form on $L^2(\Omega)$. A function $V: \Omega \rightarrow [0, \infty)$ is called form small with respect to q if there are constants $\alpha \in [0, 1)$, $\beta \in [0, \infty)$ such that*

$$\int_{\Omega} V |u|^2 dx \leq \alpha q(u) + \beta \|u\|_{L^2}^2$$

for all $u \in L^2(\Omega)$.

Definition 3.15 (Magnetic form with Dirichlet and Neumann boundary conditions): *Let $V: \Omega \rightarrow \mathbb{R}$ be measurable such that $V^+ \in L_{\text{loc}}^1(\Omega)$ and V^- is form small with respect to $\mathcal{E}_{a,0,0}^{(N)}$. Then the magnetic form with Neumann conditions is defined via*

$$D(\mathcal{E}_{a,b,V}^{(N)}) = D(\mathcal{E}_{a,b,V^+}^{(N)}), \mathcal{E}_{a,b,V}^{(N)}(u) = \sum_{j,k=1}^n \int_{\Omega} a_{jk}(D_j u) \overline{(D_k u)} dx + \int_{\Omega} V |u|^2 dx$$

and the magnetic form with Dirichlet boundary conditions $\mathcal{E}_{a,b,V}^{(D)}$ as the closure of the restriction of $\mathcal{E}_{a,b,V}^{(N)}$ to $C_c^\infty(\Omega)$.

From now on we shall always assume that $V^+ \in L^1_{\text{loc}}(\Omega)$ and that V^- is form small with respect to $\mathcal{E}_{a,0,0}^{(N)}$ for the remainder of the section.

This condition on V ensures that the form $\mathcal{E}_{a,b,V}^{(N)}$ is closed and the form $\mathcal{E}_{a,b,V}^{(D)}$ therefore closable by the KLMN Theorem (cf. [37], Thm. X.16). For a more detailed discussion we refer once again to [13].

For positive potential V , the forms without magnetic field $\mathcal{E}_{a,0,V}^{(N)}$ and $\mathcal{E}_{a,0,V}^{(D)}$ are Dirichlet forms (cf. [8], Examples 1.2.1 and 1.2.3) and the form with Dirichlet boundary conditions is regular by definition. This fact is one of the connections between the Euclidean case and the discrete case treated in Section 3.1.

Now that we have defined the Schrödinger forms, we will in a next step show that the form with magnetic field is dominated by the form without magnetic field (indeed, some variation in the potential is also allowed). To do so, we start with two technical lemmas, the first one giving some regularity results and the second one providing a product rule for weak derivatives. We omit the proof of the next lemma as it is an easy consequence of the definitions.

Lemma 3.16: (a) *If $u \in D(\mathcal{E}_{a,0,V}^{(N)})$, then $\nabla u \in L^2_{\text{loc}}(\Omega; \mathbb{C}^n)$.*
 (b) *If $u \in D(\mathcal{E}_{a,b,V}^{(N)}) \cap L^\infty(\Omega)$, then $\nabla u \in L^2_{\text{loc}}(\Omega; \mathbb{C}^n)$.*

The following Leibniz rule for weak derivatives is taken from [25], Hilfssätze 14.1.1. and 14.1.2. As the proofs are quite technical, we will not reproduce them here.

Lemma 3.17: (a) *Product rule for weak derivatives: If $u, v \in W^{1,1}_{\text{loc}}(\Omega)$ such that $u\nabla v, v\nabla u \in L^1_{\text{loc}}(\Omega; \mathbb{C}^n)$, then $uv \in W^{1,1}_{\text{loc}}(\Omega)$ and*

$$\nabla(uv) = u\nabla v + v\nabla u.$$

(b) *If $u \in W^{1,1}_{\text{loc}}(\Omega)$, then $|u| \in W^{1,1}_{\text{loc}}(\Omega)$ and*

$$|u|\nabla|u| = \text{Re}(\bar{u}\nabla u).$$

The next proposition shows that the form with magnetic potential is dominated by the form without magnetic potential. In principle, this fact has been long known and probably goes back to Simon (cf. [45]), just the regularity assumptions on a , b and V have been considerably weakened over the last decades. In the form presented here, the statement is taken from [13], Theorem 3.3. Our proof is a little different in that we prove the domination property for the forms instead of the semigroups.

Proposition 3.18: *If $\tilde{V} : \Omega \rightarrow \mathbb{R}$ is measurable, $\tilde{V} \leq V$ and \tilde{V}^- is form small with respect to $\mathcal{E}_{a,0,0}^{(N)}$, then $\mathcal{E}_{a,b,V}^{(N)}$ is dominated by $\mathcal{E}_{a,0,\tilde{V}}^{(N)}$.*

Proof. In the first step we show that $D(\mathcal{E}_{a,b,V}^{(N)})$ is a generalized ideal of $D(\mathcal{E}_{a,0,\tilde{V}}^{(N)})$.

Let $u \in D(\mathcal{E}_{a,b,V}^{(N)}) \subset W^{1,1}_{\text{loc}}(\Omega)$. By Lemma 3.17 we have $|u| \in W^{1,1}_{\text{loc}}(\Omega)$ and

$$(\dagger) \quad |u|\nabla|u| = \text{Re}(\bar{u}\nabla u) = \text{Re}(\bar{u}\nabla u - i|u|^2b) = \text{Re}(\bar{u}Du).$$

Let $\xi \in \mathbb{C}^n$ and $x \in \Omega$. Then

$$\begin{aligned} \sum_{j,k} a_{j,k}(x) \text{Re} \xi_j \text{Re} \xi_k &\leq \sum_{j,k} a_{j,k}(x) (\text{Re} \xi_j) (\text{Re} \xi_k) + \sum_{j,k} a_{j,k}(x) (\text{Im} \xi_j) (\text{Im} \xi_k) \\ &= \text{Re} \sum_{j,k} a_{j,k}(x) \xi_j \bar{\xi}_k. \end{aligned}$$

Applied to $\xi = \overline{u(x)} Du(x)$, we obtain

$$(\dagger\dagger) \quad \sum_{j,k} a_{j,k}(\partial_j|u|)(\partial_k|u|) \leq \text{Re} \sum_{j,k} a_{j,k}(D_j u)(\overline{D_k u}).$$

This implies $|u| \in D(\mathcal{E}_{a,0,V}^{(N)})$.

Next let $f_1, f_2 \in L^2(\Omega)$ such that $\nabla|f_1|, \nabla|f_2| \in L_{\text{loc}}^2(\Omega; \mathbb{C}^n)$. Assume that $\bar{f}_1 f_2 = |f_1||f_2|$ and notice that this condition is equivalent to $|f_1|f_2 = f_1|f_2|$. Differentiating this identity we obtain

$$|f_1|\nabla f_2 + f_2\nabla|f_1| = f_1\nabla|f_2| + |f_2|\nabla f_1.$$

If $f_1 \in D(\mathcal{E}_{a,b,V}^{(N)})$, $v \in D(\mathcal{E}_{a,0,V}^{(N)})^+$ such that $v \leq |f_1|$ and $f_2 = v \operatorname{sgn} f_1$, then $|f_1|, |f_2| \in D(\mathcal{E}_{a,0,V}^{(N)})$ and by Lemma 3.16 the condition $\nabla|f_1|, \nabla|f_2| \in L_{\text{loc}}^2(\Omega; \mathbb{C}^n)$ is met. Thus,

$$(\dagger \dagger \dagger) \quad |f_1|Df_2 = f_1\nabla|f_2| + |f_2|Df_1 - f_2\nabla|f_1|$$

and consequently

$$|a^{\frac{1}{2}}Df_2| \leq |a^{\frac{1}{2}}\nabla v| + |a^{\frac{1}{2}}Df_1| + |a^{\frac{1}{2}}\nabla v|.$$

Since all the summands on the right-hand side are in L^2 by assumption, so is $a^{\frac{1}{2}}Df_2$. Furthermore,

$$\int_{\Omega} V|f_2|^2 dx \leq \int_{\Omega} V|f_1|^2 dx < \infty,$$

hence $f_2 = v \operatorname{sgn} f_1 \in D(\mathcal{E}_{a,b,V}^{(N)})$.

Now let $f_1, f_2 \in D(\mathcal{E}_{a,b,V}^{(N)})$ such that $|f_1|f_2 = f_1|f_2|$. By $(\dagger \dagger \dagger)$ we have

$$|f_1|^2 \operatorname{Re} \sum_{j,k} a_{jk}(D_j f_2)(\overline{D_k f_1}) = \operatorname{Re} \sum_{j,k} a_{jk}|f_1|(\overline{D_k f_1})(f_1 \partial_j |f_2| + |f_2|D_j f_1 - f_2 \partial_j |f_1|).$$

Two applications of (\dagger) yield

$$\begin{aligned} & \operatorname{Re} \sum_{j,k} a_{jk}|f_1|(\overline{D_k f_1})(f_1 \partial_j |f_2| + |f_2|D_j f_1 - f_2 \partial_j |f_1|) \\ &= \sum_{j,k} a_{jk}|f_1|^2(\partial_j |f_2|)(\partial_k |f_1|) + \operatorname{Re}|f_1||f_2| \sum_{j,k} a_{jk}(D_j f_1)(\overline{D_k f_1}) \\ & \quad - \operatorname{Re} \sum_{j,k} a_{jk}|f_2|f_1(\overline{D_k f_1})\partial_j |f_1| \\ &= |f_1|^2 \sum_{j,k} a_{jk}(\partial_j |f_2|)(\partial_k |f_1|) + |f_1||f_2| \operatorname{Re} \sum_{j,k} a_{jk}(D_j f_1)(\overline{D_k f_1}) \\ & \quad - |f_1||f_2| \sum_{j,k} a_{jk}(\partial_j |f_1|)(\partial_k |f_1|). \end{aligned}$$

Using $(\dagger \dagger)$, we conclude

$$\begin{aligned} & |f_1|^2 \operatorname{Re} \sum_{j,k} a_{jk}(D_j f_2)(\overline{D_k f_1}) \\ &= |f_1|^2 \sum_{j,k} a_{jk}(\partial_j |f_2|)(\partial_k |f_1|) + |f_1||f_2| \operatorname{Re} \sum_{j,k} a_{jk}(D_j f_1)(\overline{D_k f_1}) \\ & \quad - |f_1||f_2| \sum_{j,k} a_{jk}(\partial_j |f_1|)(\partial_k |f_1|) \\ &\geq |f_1|^2 \sum_{j,k} a_{jk}(\partial_j |f_2|)(\partial_k |f_1|). \end{aligned}$$

Therefore,

$$\begin{aligned} \operatorname{Re} \mathcal{E}_{a,b,V}^{(N)}(f_1, f_2) &= \operatorname{Re} \mathcal{E}_{a,b,0}^{(N)}(f_1, f_2) + \int_{\Omega} V f_1 \overline{f_2} dx \\ &\geq \mathcal{E}_{a,0,0}^{(N)}(S(f_1), S(f_2)) + \int_{\Omega} \tilde{V} |f_1||f_2| dx \\ &= \mathcal{E}_{a,0,\tilde{V}}^{(N)}(|f_1|, |f_2|). \end{aligned}$$

□

Remark: Notice that while domination is characterized by an inequality for integrals, our proof gives a *pointwise* inequality generalizing the diamagnetic inequality (cf. [23], Theorem 7.21)

$$|Du| \geq |\nabla|u||.$$

In particular, the regularity requirements are only needed to assure that Lemma 3.17 is applicable, the rest of the proof is purely algebraic in means.

Having proven the domination, we will now show that $\mathcal{E}_{a,b,V}^{(N)}$ is smoothly inner regular so that we can apply Corollary 2.5.

Proposition 3.19: *The form $\mathcal{E}_{a,b,V}^{(N)}$ is smoothly inner regular.*

Proof. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $\text{supp } \eta \subset \overline{B_1(0)}$, $\eta \geq 0$ and $\|\eta\|_{L^1} = 1$, and define $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(\varepsilon^{-1}x)$.

Let $u \in D(\mathcal{E}_{a,b,V}^{(N)}) \cap L_c^\infty(\Omega)$ and $K \subset\subset \Omega$ be open such that $u|_{K^c} = 0$. Let

$$K_\varepsilon = \{x \in \Omega \mid d(x, K) < \varepsilon\}$$

for $\varepsilon < d(K, \Omega^c)$ and $\delta = \frac{1}{2}d(K, \Omega^c)$.

By Lemma 3.16, $a^{\frac{1}{2}}\nabla u \in L_{\text{loc}}^2(\Omega; \mathbb{C}^n)$. Since $u|_{K^c} = 0$, we have that $\|a^{\frac{1}{2}}\nabla u\|_{L^2(\Omega)} = \|a^{\frac{1}{2}}\nabla u\|_{L^2(K_\delta)} < \infty$. Moreover, $a \geq \mu_{K_\delta}$ on K_δ implies $a^{-\frac{1}{2}} \leq \mu_{K_\delta}^{-\frac{1}{2}}$ on K_δ , hence $\nabla u = a^{-\frac{1}{2}}a^{\frac{1}{2}}\nabla u \in L^2(K_\delta)$. Since ∇u vanishes outside K_δ , we have $u \in H^1(\Omega)$. Extend u by 0 to \mathbb{R}^n and define $u_\varepsilon = (u * \eta_\varepsilon)|_\Omega$ for $\varepsilon < \delta$. Then $u_\varepsilon \in C_c^\infty(\Omega)$, $\text{supp } u_\varepsilon \subset K_\varepsilon \subset K_\delta$ and $u_\varepsilon \rightarrow u, \varepsilon \downarrow 0$, in $H^1(\Omega)$ and pointwise by the mollifier theorem (cf. [23]). Moreover, $|u_\varepsilon(x)| \leq \|u\|_\infty$ for all $x \in \Omega$.

Since $a \in L^\infty(K_\delta; M_n(\mathbb{R})^+)$, we have

$$\|a^{\frac{1}{2}}(\nabla u - \nabla u_\varepsilon)\|_{L^2(K_\delta)} \leq \|a^{\frac{1}{2}}\|_{L^\infty(K_\delta)} \|u - u_\varepsilon\|_{H^1(K_\delta)} \rightarrow 0, \varepsilon \downarrow 0.$$

Since $a \in L^\infty(K_\delta; M_n(\mathbb{R})^+)$, $b \in L^2(K_\delta; \mathbb{R}^n)$ and $V^+ \in L^1(K_\delta)$ and $|u_\varepsilon(x)| \leq \|u\|_\infty$ for almost all $x \in \Omega$, an application of the dominated convergence theorem yields

$$\begin{aligned} \|(u - u_\varepsilon)a^{\frac{1}{2}}b\|_{L^2}^2 &= \int_{K_\delta} |(u_\varepsilon - u)a^{\frac{1}{2}}b|^2 dx \rightarrow 0, \varepsilon \downarrow 0, \\ \|(V^+)^{\frac{1}{2}}|u - u_\varepsilon|\|_{L^2}^2 &= \int_{K_\delta} V^+ |u - u_\varepsilon|^2 dx \rightarrow 0, \varepsilon \downarrow 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathcal{E}_{a,b,V^+}^{(N)}(u - u_\varepsilon) &= \|a^{\frac{1}{2}}D(u - u_\varepsilon)\|_{L^2}^2 + \|(V^+)^{\frac{1}{2}}(u - u_\varepsilon)\|_{L^2}^2 \\ &\leq 2\|a^{\frac{1}{2}}\nabla(u - u_\varepsilon)\|_{L^2}^2 + \|(u - u_\varepsilon)a^{\frac{1}{2}}b\|_{L^2}^2 + \|(V^+)^{\frac{1}{2}}|u - u_\varepsilon|\|_{L^2}^2 \\ &\rightarrow 0, \varepsilon \downarrow 0. \end{aligned}$$

Finally, since V^- is form small with respect to $\mathcal{E}_{a,b,V^+}^{(N)}$, we also have $\mathcal{E}_{a,b,V}^{(N)}(u - u_\varepsilon) \rightarrow 0, \varepsilon \downarrow 0$.

Thus, $D(\mathcal{E}_{a,b,V}^{(N)}) \cap L_c^\infty(\Omega) \subset \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{a,b,V}} = D(\mathcal{E}_{a,b,V}^{(D)})$. \square

The main theorem of this section is now an immediate consequence of the abstract results in Chapter 2 and the preceding propositions.

Theorem 3.20: *Let $\tilde{V}: \Omega \rightarrow \mathbb{R}^n$ be measurable such that $\tilde{V} \leq V$ and \tilde{V}^- is relatively form bounded with respect to $\mathcal{E}_{a,0,\tilde{V}}^{(D)}$. If $\mathcal{E}_{a,0,\tilde{V}}^{(D)} = \mathcal{E}_{a,0,\tilde{V}}^{(N)}$, then $\mathcal{E}_{a,b,V}^{(D)} = \mathcal{E}_{a,b,V}^{(N)}$.*

Proof. By Proposition 3.18, $\mathcal{E}_{a,b,V}^{(N)}$ is dominated by $\mathcal{E}_{a,0,\tilde{V}}^{(N)}$. By Proposition 3.19, the form $\mathcal{E}_{a,b,V}^{(N)}$ is smoothly inner regular. Thus, an application of Corollary 2.5 yields $\mathcal{E}_{a,b,V}^{(N)} = \mathcal{E}_{a,b,V}^{(D)}$. \square

Lemma 3.21: *If $\mathcal{E}_{a,b,V^+}^{(D)} = \mathcal{E}_{a,b,V^+}^{(N)}$, then $\mathcal{E}_{a,b,V}^{(D)} = \mathcal{E}_{a,b,V}^{(N)}$.*

Proof. Let $u \in D(\mathcal{E}_{a,b,V}^{(N)}) = D(\mathcal{E}_{a,b,V^+}^{(N)}) = D(\mathcal{E}_{a,b,V^+}^{(D)})$. By assumption there is a sequence $\varphi_n \in C_c^\infty(\Omega)$ such that $\|u - \varphi_n\|_{a,b,V^+} \rightarrow 0$. But

$$\|u - \varphi_n\|_{a,b,V}^2 = \|u - \varphi_n\|_{a,b,V^+}^2 - \int_{\Omega} V^- |u - \varphi_n|^2 dx \leq \|u - \varphi_n\|_{a,b,V^+}^2.$$

Hence $C_c^\infty(\Omega)$ is also dense in $D(\mathcal{E}_{a,b,V}^{(N)})$, that is, $\mathcal{E}_{a,b,V}^{(D)} = \mathcal{E}_{a,b,V}^{(N)}$. \square

Corollary 3.22: *If $\mathcal{E}_{a,0,0}^{(N)} = \mathcal{E}_{a,0,0}^{(D)}$, then $\mathcal{E}_{a,b,V}^{(N)} = \mathcal{E}_{a,b,V}^{(D)}$.*

Proof. Just combine Theorem 3.20 and Lemma 3.21. \square

We conclude the section giving some examples in which the minimal and maximal Schrödinger form without magnetic field coincide and in which our theorem is thus applicable.

Example 3.23: If $\Omega = \mathbb{R}^n$, then $-\Delta$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^n)$ and therefore $\mathcal{E}_{1,0,0}^{(N)} = \mathcal{E}_{1,0,0}^{(D)}$. This is a classical result in the theory of elliptic differential operators (see [46], Satz 11.15, for example).

This example is a nice illustration of the strength of our method: From this comparably simple result concerning an elliptic differential operator with constant coefficients we can infer $\mathcal{E}_{1,b,V}^{(N)} = \mathcal{E}_{1,b,V}^{(D)}$ under the weakest possible regularity assumptions on b and V .

For divergence type forms without potential a characterization of $\mathcal{E}^{(N)} = \mathcal{E}^{(D)}$ is given in [38], Prop. 4.1. combined with [39], Prop. 4.5. (also compare [9] for analogous results on weighted manifolds) in terms of the capacity of the boundary: For a measurable set $A \subset \overline{\Omega}$ the capacity is defined as

$$\text{cap}(A) = \inf\{\|u\|_{a,0,0}^2 \mid u \in D(\mathcal{E}_{a,0,0}^{(N)}), \text{ there is } U \supset A \text{ open: } u|_{U \cap \Omega} = 1\}.$$

Example 3.24: Assume that $a_{jk} \in W^{1,\infty}(\Omega)$ for $1 \leq j, k \leq n$ (we believe that the proofs in [38, 39] carry over to the case $a_{jk} \in L^\infty(\Omega)$). Then $\mathcal{E}_{a,0,0}^{(N)} = \mathcal{E}_{a,0,0}^{(D)}$ if and only if $\text{cap}(\partial\Omega) = 0$.

For a bounded domain Ω the (non-magnetic) forms with Dirichlet and Neumann boundary conditions coincide if the potential grows sufficiently fast at the boundary $\partial\Omega$. An estimate in the one-dimensional case follows from Thm. X.10 in [37].

Example 3.25: Assume that $V \in C((0,1))$ is positive and

$$V(x) \geq \frac{3}{4} \frac{1}{d(x)^2}$$

near 0 and 1, where $d(x) = \min\{x, 1-x\}$. Then $-\Delta + V$ is essentially self-adjoint on $C_c^\infty((0,1))$ and consequently $\mathcal{E}_{1,0,V}^{(D)} = \mathcal{E}_{1,0,V}^{(N)}$.

This result has been generalized to more than one dimension and improved in various directions. We just mention as one example the conditions given in Thm. 2 of [33]. That $\mathcal{E}_{1,b,V}^{(D)} = \mathcal{E}_{1,b,V}^{(N)}$ holds under these conditions seems to be new.

Example 3.26: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 -boundary. Assume that $V = V_1 + V_2$, $V_1 \in L^\infty(\Omega)$ and

$$V_2(x) \geq \frac{1}{d(x, \partial\Omega)^2} \left(\frac{3}{4} - \frac{1}{\ln d(x, \partial\Omega)^{-1}} - \frac{1}{\ln d(x, \partial\Omega)^{-1} \ln \ln d(x, \partial\Omega)^{-1}} - \dots \right)$$

for all $x \in \Omega$ near $\partial\Omega$. Then $-\Delta + V$ is essentially self-adjoint on $C_c^\infty(\Omega)$ and $\mathcal{E}_{1,0,V}^{(D)} = \mathcal{E}_{1,0,V}^{(N)}$ follows.

REFERENCES

- [1] C. Anné and N. Torki-Hamza. The Gauss-Bonnet operator of an infinite graph. *Anal. Math. Phys.*, 5(2):137–159, 2015.
- [2] P. H. Bérard. *Spectral Geometry: Direct and Inverse Problems*. Lecture Notes in Mathematics. Springer-Verlag, 1986.
- [3] M. Braverman, O. Milatovic, and M. Shubin. Essential self-adjointness of schrödinger-type operators on manifolds. *Russian Math. Surveys*, 57(4):641, 2002.
- [4] Y. Colin de Verdière, N. Torki-Hamza, and F. Truc. Essential self-adjointness for combinatorial Schrödinger operators II—metrically non complete graphs. *Math. Phys. Anal. Geom.*, 14(1):21–38, 2011.
- [5] Y. Colin de Verdière, N. Torki-Hamza, and F. Truc. Essential self-adjointness for combinatorial Schrödinger operators III—Magnetic fields. *Ann. Fac. Sci. Toulouse Math. (6)*, 20(3):599–611, 2011.
- [6] A. Eberle. *Uniqueness and non-uniqueness of semigroups generated by singular diffusion operators*. Lecture Notes in Mathematics, 1718. Springer-Verlag, 1999.
- [7] R. L. Frank, D. Lenz, and D. Wingert. Intrinsic metrics for non-local symmetric Dirichlet forms and applications to spectral theory. *J. Funct. Anal.*, 266(8):4765–4808, 2014.
- [8] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet Forms and Symmetric Markov Processes*. De Gruyter Studies in Mathematics Series. De Gruyter, 1994.
- [9] A. Grigor'yan and J. Masamune. Parabolicity and stochastic completeness of manifolds in terms of the Green formula. *J. Math. Pures Appl. (9)*, 100(5):607–632, 2013.
- [10] S. Haeseler, M. Keller, D. Lenz, and R. Wojciechowski. Laplacians on infinite graphs: Dirichlet and Neumann boundary conditions. *J. Spectr. Theory*, 2:397–432, 2012.
- [11] X. Huang, M. Keller, J. Masamune, and R. Wojciechowski. A note on self-adjoint extensions of the Laplacian on weighted graphs. *J. Funct. Anal.*, 265(8):1556–1578, 2013.
- [12] B. Hua and Y. Lin. Stochastic completeness for graphs with curvature dimension conditions. *arXiv preprint arXiv:1504.00080*, 2015.
- [13] D. Hundertmark and B. Simon. A diamagnetic inequality for semigroup differences. *J. Reine Angew. Math.*, pages 107–130, 2004.
- [14] H. Hess, R. Schrader, and D. A. Uhlenbrock. Domination of semigroups and generalization of Kato's inequality. *Duke Math. J.*, 44(4):893–904, 1977.
- [15] G. Isac and A. B. Németh. Every generating isotone projection cone is latticial and correct. *J. Math. Anal. Appl.*, 147(1):53–62, 1990.
- [16] T. Kato. Schrödinger operators with singular potentials. *Israel J. Math.*, 13(1-2):135–148, 1972.
- [17] T. Kawabata and M. Takeda. On uniqueness problem for local Dirichlet forms. *Osaka J. Math.*, 33(4):881–893, 1996.
- [18] M. Keller and D. Lenz. Unbounded Laplacians on graphs: basic spectral properties and the heat equation. *Math. Model. Nat. Phenom.*, 5(04):198–224, 2010.
- [19] M. Keller and D. Lenz. Dirichlet forms and stochastic completeness of graphs and subgraphs. *J. Reine Angew. Math.*, 666:189–223, 2012.
- [20] K. Kuwae. Reflected Dirichlet forms and the uniqueness of Silverstein's extension. *Potential Anal.*, 16(3):221–247, 2002.
- [21] K. Kuwae and T. Shioya. Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry. *Comm. Anal. Geom.*, 11(4):599–673, 2003.
- [22] K. Kuwae and Y. Shiozawa. A remark on the uniqueness of Silverstein extensions of symmetric Dirichlet forms. *Math. Nachr.*, 288(4):389–401, 2015.
- [23] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.
- [24] P. Li and G. Tian. On the heat kernel of the Bergmann metric on algebraic varieties. *J. Amer. Math. Soc.*, 8(4):857–877, 1995.
- [25] A. Manavi. *Zur Störung von dominierten C_0 -Halbgruppen auf Banachfunktionsräumen mit ordnungstetiger Norm und sektoriellen Formen mit singulären komplexen Potentialen*. PhD thesis, TU Dresden, 2001.
- [26] J. Masamune. Essential self-adjointness of Laplacians on Riemannian manifolds with fractal boundary. *Comm. Partial Differential Equations*, 24(3-4):749–757, 1999.
- [27] J.-J. Moreau. Décomposition orthogonale d'un espace hilbertien selon deux cônes mutuellement polaires. *C. R. Acad. Sci. Paris*, 255:238–240, 1962.
- [28] J. W. Milnor and J. D. Stasheff. *Characteristic classes*. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 76.
- [29] O. Milatovic and F. Truc. Self-adjoint extensions of discrete magnetic Schrödinger operators. *Ann. Henri Poincaré*, 15(5):917–936, 2014.
- [30] O. Milatovic and F. Truc. Maximal accretive extensions of Schrödinger operators on vector bundles over infinite graphs. *Integral Equations Operator Theory*, 81(1):35–52, 2015.
- [31] A. Manavi, H. Vogt, and J. Voigt. Domination of semigroups associated with sectorial forms. *J. Operator Theory*, 54(1):9–26, 2005.

- [32] A. B. Németh. Characterization of a Hilbert vector lattice by the metric projection onto its positive cone. *J. Approx. Theory*, 123(2):295–299, 2003.
- [33] G. Nenciu and I. Nenciu. On confining potentials and essential self-adjointness for Schrödinger operators on bounded domains in \mathbb{R}^n . *Ann. Henri Poincaré*, 10(2):377–394, 2009.
- [34] E. Ouhabaz. Invariance of closed convex sets and domination criteria for semigroups. *Potential Anal.*, 5(6):611–625, 1996.
- [35] E. Ouhabaz. L^p contraction semigroups for vector valued functions. *Positivity*, 3(1):83–93, 1999.
- [36] R. C. Penney. Self-dual cones in Hilbert space. *J. Funct. Anal.*, 21(3):305–315, 1976.
- [37] M. Reed and B. Simon. *Methods of modern mathematical physics. 2. Fourier analysis, self-adjointness*, volume 2. Elsevier, 1975.
- [38] D. W. Robinson and A. Sikora. Markov uniqueness of degenerate elliptic operators. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 10(3):683–710, 2011.
- [39] D. W. Robinson. Uniqueness of diffusion operators and capacity estimates. *J. Evol. Equ.*, 13(1):229–250, 2013.
- [40] M. Schmidt. Energy forms. Dissertation, draft version, 2016.
- [41] M. Shubin. Essential self-adjointness for semi-bounded magnetic Schrödinger operators on non-compact manifolds. *J. Funct. Anal.*, 186(1):92–116, 2001.
- [42] M. L. Silverstein. Symmetric Markov processes. *Lecture Notes in Mathematics*, Vol. 426. Springer-Verlag, 1974.
- [43] B. Simon. An Abstract Kato’s Inequality for Generators of Positivity Preserving Semigroups. *Indiana Univ. Math. J.*, 26(6), 1977.
- [44] B. Simon. Kato’s inequality and the comparison of semigroups. *J. Funct. Anal.*, 32(1):97–101, 1979.
- [45] B. Simon. Maximal and minimal Schrödinger forms. *J. Operator Theory*, 1:37–47, 1979.
- [46] J. Weidmann. *Lineare Operatoren in Hilberträumen. Teil I Grundlagen*. B. G. Teubner, 2 edition, 2000.
- [47] D. Werner. *Funktionalanalysis*. Springer-Verlag, Berlin, extended edition, 2007.
- [48] R. K. Wojciechowski. *Stochastic completeness of graphs*. PhD thesis, City University of New York, 2008.

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